Some Applications of Differentiation – Single Variable Case

In economics the differential calculus has had many prolific applications. It is convenient at this stage to list some of the functional relationships which recur most frequently in the work of the economists:

- A production function Q= f(L) which records the maximum amount of output that can be produced with given amount of labour.
- A cost function C = f(Q)records the total expenses C associated with production level Q.
- A utility function U(Q), which measures the pleasure that the individual derives from the ownership of some quantity of Q of some commodity.
- A revenue function P.Q= Q.F(Q), which shows the total income of the firm when it sells Q units of a commodity at the price P per unit;

Economists have then adopted the following terminology:

Marginal product is the name given to $\frac{dQ}{dL}$

Marginal cost refers to $\frac{dC}{dQ}$

Marginal utility refers to $\frac{dU}{dQ}$

Marginal revenue refers to $\frac{d(P.Q)}{dQ}$

Example 3. (a) For the total revenue function $TR = 500q - 2q^2$, Find the value of MR when q= 20

(b) If $P = 80 - 4q$ is the linear demand function, write out the total revenue and hence the marginal revenue functions

Solution:

(a) $MR = \frac{d(TR)}{dq} = 500 - 4q$

Thus when q = 20

MR = 500 – 80 = 420

(b) We know by definition that TR = pq.

$$\Rightarrow \quad TR = (80 - 4q)\, q = 80q - 4q^2$$

$$MR = \frac{d(TR)}{dq} = 80 - 8q$$

Example 4. Given the following total cost function, determine the level of output thatminimises the average cost and marginal cost:

$$TC = q^3 - 24q^2 + 600q$$

Solution: convert the total cost function into average cost by dividing by q

$$AC = q^2 - 24q + 600$$

Now to find the minimum of the average cost function, set the first derivative of AC function equal to zero.

$$\frac{d(AC)}{dq} = 2q - 24 = 0 \Rightarrow q = 12$$

At $q = 12$, the average cost function reaches its optima. The next step is to take the second derivative of the average cost function to determine whether q=12 is its minimum or not.

$$\frac{d^2(AC)}{dq} = 2 > 0$$

Since the second derivative of the average cost function is positive, it confirms that the function is minimum at q=12

Now for the marginal cost function, following the same analogy

$$\frac{d(TC)}{dq} = MC = 3q^2 - 48q + 600$$

To find the minimum point on the MC curve, set the first derivative of the marginal cost function equal to zero

$$MC' = 6q - 48 = 0 \Rightarrow q = 8$$

At q=8, the MC function reaches its optima. The next step is to take the second derivative of the MC' to determine whether q = 8 is its minimum or not

$$MC'' = 6 > 0$$

Since the second derivative of the MC function is positive this implies that the MC function is minimum at q= 8.

Example 5. If C(x) is the total cost function, then using calculus show that at AC =MC at the minimum point of AC.

Solution. By definition AC = $\frac{C(x)}{x}$

For the minimum value of the AC, we first set the first derivative equal to zero

$$AC' = \frac{X(C'(X)-C(X)}{X^2} = 0 \Rightarrow AC' = C'(X) - \frac{C(X)}{X} = 0 \Rightarrow C'(X) = \frac{C(X)}{X} \Rightarrow MC = AC$$

Thus MC = AC only at minimum value of AC.

3.3.1. Profit Maximisation of a Firm

We will now see how calculus can help a firm maximise its profit. From the very elementary economics, the students are well acquainted with the marginal cost equals marginal revenue as a perquisite for profit maximisation. Let us see the derivation of this condition. If R(Q) is the revenue function of a0 firm and C(Q) the cost function. From these it follows that a profit function $\boldsymbol{\pi}$ may be formulae as

$$\pi = R(Q) - C(Q) \quad \text{.................................} \quad \text{(i)}$$

A firm sets its output where its marginal profit is zero.

We obtain this result formally using first order condition for a profit maximisation. We set the first derivative of the profit function, equation (i) with respect to quantity equal to zero.

$$\frac{d\pi}{dQ} = 0 \quad \text{..} \quad \text{(ii)}$$

Equation (ii) is a necessary condition for profit to be maximised. Sufficiency requires, in addition, that the second order condition hold:

$$\frac{d^2\pi}{dQ^2} < 0 \quad \text{...} \quad \text{(iii)}$$

Because profit is a function of revenue and cost, we can state the above in one additional way. The first order condition can be stated by setting the first order derivative of $\pi = R(Q) - C(Q)$ equal to zero.

$$\frac{d\pi}{dQ} = \frac{dR}{dQ} - \frac{dC}{dQ} = MR - MC = 0$$

$$\Rightarrow MR = MC \quad \text{...} \quad \text{(iv)}$$

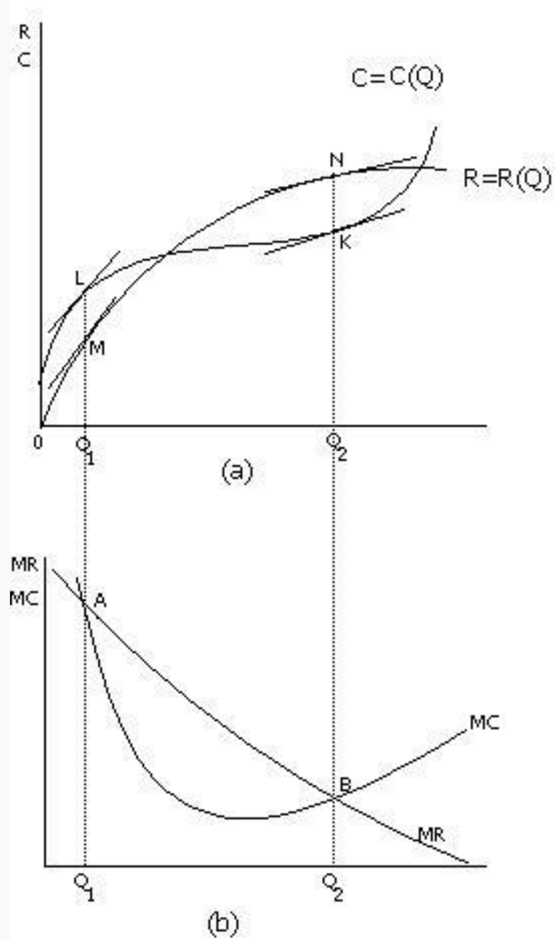
Equation (iv) is a first order condition which states that for profit to be maximised, the marginal cost must be equal to marginal revenue of the output.
For profit to be maximised, the second order condition must hold.

$$\frac{d^2\pi}{dQ^2} = \frac{d^2R}{dQ^2} - \frac{d^2C}{dQ^2} = \frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} < 0$$

That is, for profit to be maximised, the slope of the marginal revenue curve, $d(MR)/dQ$ must be less than the slope of the marginal cost, $(d(MC))/dQ$ curve.

The above conditions are illustrated in Fig.2.9. in Fig.2.9(a) we have drawn the total cost and revenue curves and in Fig 2.9(b) the marginal curves are drawn.

FIG.2.9



The necessary condition for profit maximisation is MR =MC.But as the diagram shows, we have two points A and B.which one should it be ? at point A the slope of the MR curve is numerically smaller than the slope of the marginal revenue curve i.e. $\frac{d(MR)}{dQ} > \frac{d(MC)}{dQ}$.thus point A or the out put level Q_1 violates the sufficient condition for relative maximum. It is point B or the output level Q_3 where both necessary as well as sufficient conditions are satisfied. [1]

[1] In general TR= p(Q).Q

MR = $\frac{d(TR)}{dQ} = \frac{dp(Q).Q}{dQ} = \frac{p(Q)dQ}{dQ} + \frac{dp(Q)}{dQ}.Q$

Example 6. Given the total cost function $C = 5q + \frac{q^2}{50}$ and the demand function q = 400 -20p.

(a) Find the total revenue function
(b) Maximise the total revenue function
(c) Maximise profit function

Solution: (i) We know that

$$TR = p.q$$

Let us first change the demand function

$$p = 20 - \frac{q}{20}$$

Then $TR = \left(20 - \frac{q}{20}\right) q = 20q - \frac{q^2}{20}$

(ii) In order to maximise the total revenue function, find the critical value(s) by setting set the first derivative of TR function equal to zero

$$\frac{d(TR)}{dq} = 20 - \frac{q}{10} = 0$$

And we have $q = 200$

At critical value the total revenue function is maximised provided the sufficient condition is satisfied

That is $\frac{d^2(TR)}{dq^2} < 0$

$$\frac{d^2(TR)}{dq^2} = -\frac{1}{10} < 0$$

Thus total revenue is maximised at $q = 200$ and the maximum total revenue is

$TR = 20q - \frac{q^2}{20} = 20(200) - \frac{(200)^2}{20} = 4000 - 2000 = 2000$

(i) The profit function of the firm is

$$\pi = TR - TC = 15q - \frac{7q^2}{100}$$

In the above equation, the first term on the right side is the price or average revenue. the second term is the slope of the demand curve $\frac{dp(Q)}{dQ}$ times the number of units sold. Under monopoly MR<P because $\frac{dp(Q)}{dQ} < 0$ while under perfect competition MR=P=AR

The first order condition for profit maximisation is

$$\frac{d\pi}{dq} = \frac{d(TR)}{dq} - \frac{d(TC)}{dq} = 0$$

$$= 20 - \frac{q}{10} - 5 - \frac{q}{25} = 0$$

$$\Rightarrow q = 107 \text{ (Approximately)}$$

Whether profit are maximised at $q = 107$, we need to ensure that the sufficient condition is satisfied

That is $\frac{d^2\pi}{dq^2} < 0$

$$\frac{d^2\pi}{dq^2} = \frac{d^2(TR)}{dq^2} - \frac{d^2(C)}{dq^2} = -\frac{1}{10} - \frac{1}{25} < 0$$

Thus profit are maximised at $q = 107$ and the maximum profit are $\pi = 803$

3.3.2. Short run production analysis

Short run is the production period in which the input L is allowed to vary by keeping K constant. Thus the general format of the short run production function is written as $Q = f(L, \overline{K})$, where $\overline{K}$ denotes the constant amount of the capital. The short run production function in its cubic form is drawn in the fig 2.10 (a) and the corresponding marginal and average productivities are shown in 2.10 (b). the Marginal product curve cuts the average product curve at its maximum. We can show this result mathematically.

Let $Q = f(L, \overline{K})$

The average product for labour would be (keeping capital K constant)

$$\frac{Q}{L} = \frac{f(L, \overline{K})}{L}$$

The first order maximisation condition for the average product of L is,

$$\frac{d(\frac{Q}{L})}{dL} = \frac{L.f'(L,\overline{K}) - f(L,\overline{K})}{L^2} = 0 \qquad \text{......................} \quad \text{(i)}$$

Provided $\frac{d^2(\frac{Q}{L})}{dL} < 0$

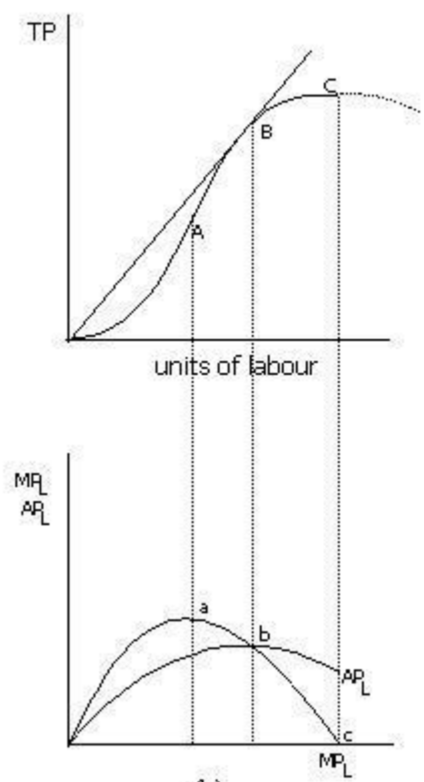
From (i) we have

$$f'(L, \overline{K}) = \frac{f(L,\overline{K})}{L}$$

Or $MP_L = AP_L$

From above it is obvious that marginal product curve hits average product curve when average product is at its maximum. One more point which is significant in the figure 2.10 (b) is that marginal product reaches maximum at a smaller input of labour.

Fig.2.10 (a)



(b)

Example 7. Consider a production function given by the sixth degree equation

$Q = AL^2K^2 - BL^3K^3$ where A,B >0 find

(a) Average product of labour
(b) Marginal product of labour
(c) Show that $AP_L = MP_L$ at the maximum of AP_L

(d) Show that MP_L reaches maximum at lower input level than AP_L

Solution:

Letting $AK^2 = X_1$ and $BK^3 = X_2$ so that the production function is now given as

$$Q = X_1 L^2 - X_2 L^3$$

Now by definition

$AP_L = X_1 L - X_2 L^2$ and $MP_L = 2X_1 L - 3X_2 L^2$

Now for the maximisation of the AP_L, set the first derivative of $AP_L = 0$

i.e. $AP_L' = X_1 - 2X_2 L = 0 \Rightarrow L = \frac{X_1}{2X_2}$

the second order condition for the maximisation is $AP_L'' < 0$

$AP_L'' = -2X_2 < 0$

Thus AP_L is maximised at $L = \frac{X_1}{2X_2}$ and the maximum value is

$$(AP_L)_{L = \frac{X_1}{2X_2}} = \frac{X_1^2}{4X_2^2}$$

At this point the value of MP_L is obtained simply by substituting the value $L = \frac{X_1}{2X_2}$ in MP_L function

$$(MP_L)_{L = \frac{X_1}{2X_2}} = 2X_1 . \frac{X_1}{2X_2} - 3X_2 . \left(\frac{X_1}{2X_2}\right)^2$$

$$(MP_L)_{L = \frac{X_1}{2X_2}} = \frac{X_1^2}{4X_2^2} = (AP_L)_{L = \frac{X_1}{2X_2}}$$

Thus we see that $AP_L = MP_L$ at the maximum of AP_L

Now, for the maximisation of MP_L we follow the same procedure

Set the first derivative of the MP_L equal to zero

$$MP_L' = 2X_1 - 6X_2 L = 0 \Rightarrow L = \frac{X_1}{3X_2}$$

The second order condition for the maximisation is $MP_L'' < 0$

$MP_L'' = -6X_2 < 0$

Thus MP_L is maximised at $L = \frac{X_1}{3X_2}$ and the maximum value is

$$(MP_L)_{L = \frac{X_1}{3X_2}} = \frac{{X_1}^2}{3X_2}$$

AP reaches maximum at $L = \frac{X_1}{2X_2}$ and MP reaches maximum at $L = \frac{X_1}{3X_2}$. Since $X_1, X_2, L > 0$, MP reaches its maximum at a smaller input of labour than AP.