Rules of Differentiation

The process of finding the derivative of a function is called Differentiation.[1] In the previous chapter, the required derivative of a function is worked out by taking the limit of the difference quotient. It would be tedious, however, to have to do this every time we wanted to find the derivative of a function, for there are various rules of differentiation that will enable us to find the derivative of a desired function directly. Students are advised to equip themselves with the following rules to be able to apply them in the subsequent topics like marginal analysis, optimisation (Unconstrained and Constrained) problems.

Following are some of the rules of Differentiation.

1. **Constant Function Rule**

   The derivative of a constant function $f(x) = A$, where A is a constant, is zero.

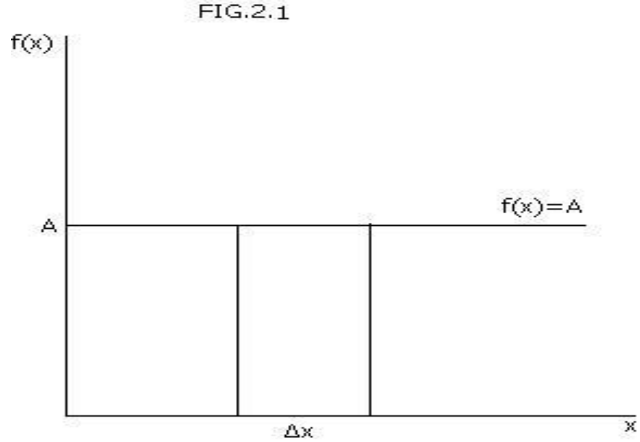
   I.e. if $f(x) = A$ then $f'(x) = 0$

The reason for $f'(x) = 0$ for $f(x) = A$ is easy to see intuitively by having a look at the graph (Fig 2.1) of a constant function. The graph is a horizontal straight line with a zero Slope throughout.

Example 2.1. Given $f(x) = -5$ $\qquad f'(x) = 0$

$f(x) = 10 \qquad f'(x) = 0$

FIG.2.1



[1] Differentiation is an operation that transforms a function f into another function f'.

2. **Linear Function Rule**

The derivative of a linear function $f(x) = a + bx$ with a and b constants is equal to b. For example the derivative of a function $f(x) = 3 + 2x$ is 2.it is obvious that the derivative of a linear function is the multiplicative constant of the variable. The important implication of this result is that for a linear function the rate at which variable y changes with respect to a change in x is same at every value of x.

Example 2.2. Find the derivative of

a. $q = 10 - 2p$; where q is quantity demanded and p the price

b. $C = 5 + \frac{1}{2}Y$; where C is consumption , Y is income

Solution **a.** Given $q = 3 - 2p \quad \Rightarrow \quad \frac{dq}{dp} = -2$ means that quantity demanded falls by two units for every one unit increase in the price (see Fig. 2.2) .

a. Given $C = 5 + \frac{1}{2}Y \quad \Rightarrow \frac{dc}{dy} = \frac{1}{2}$

The derivative $\frac{dc}{dy}$ or C' is called marginal propensity to consume (mpc). In the above example the value of mpc = 0.5 > 0 . the graphical exposition of the function is shown in fig.2.3.

FIG 2.2

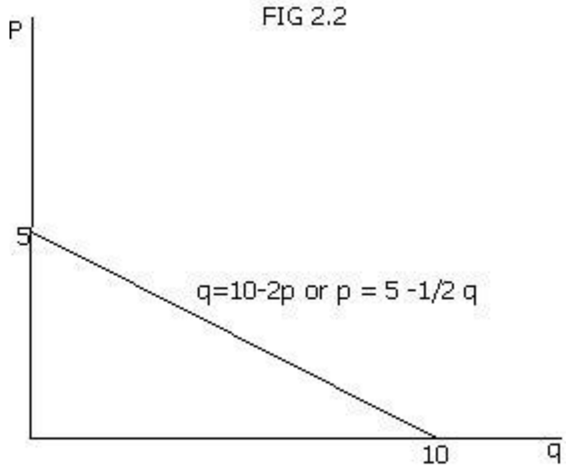
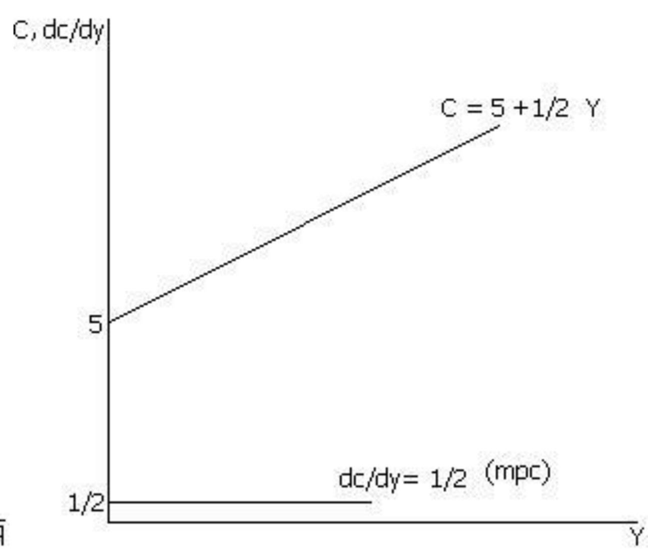


FIG 2.3



3. **Power Function Rule.**

The derivative of a function $f(x) = x^n$; where n is any arbitrary constant is n multiplied by the variable raised to the power $n - 1$.

i.e if $f(x) = x^n$ then

$$f'(x) = nx^{n-1}.$$

For $n = 2$, the rule was already confirmed in the example 10 of the previous chapter.

Few remarks about the power function $f(x) = x^n$

- If $n > 1$; the derivative of the function will be increasingfunction of x
- If $n = 1$; the drivative of the function will remain constant function and will be equal to 1
- If $n < 1$; the derivative of the function will be decreasing function of x .

Example 2.3. find the derivative of the following functions given below and draw there respective Graphs to illustrate the remarks about power function.

a. $f(x) = x^{\frac{3}{2}}$ **b.** $f(x) = x$ **c.** $f(x) = x^{\frac{1}{2}}$

Solution. a). Given $f(x) = x^{\frac{3}{2}}$ $\Rightarrow f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}}$

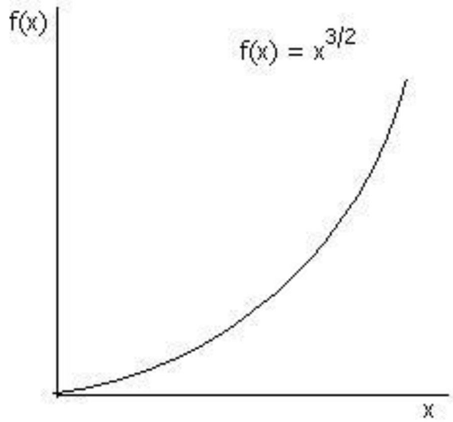
b) Given $f(x) = x$ $\Rightarrow f'(x) = 1$ $[\because x = x^1]$

c) Given $f(x) = x^{\frac{1}{2}}$ $\Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

the diagramatic exposition of the above functions is shown in Fig. 2.4.[2]

[2] If x is considered as Labour input, then the three graphs portrays increasing, constant and decreasing marginal physical productivity of Labour.

FIG 2.4 (a)



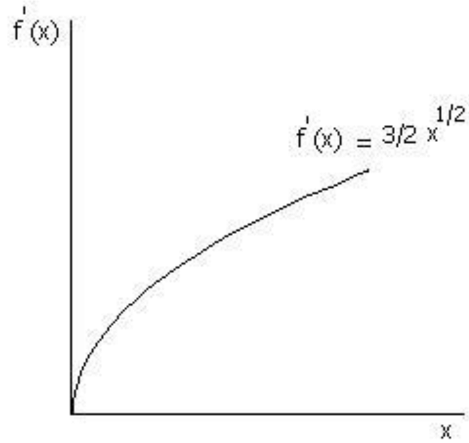
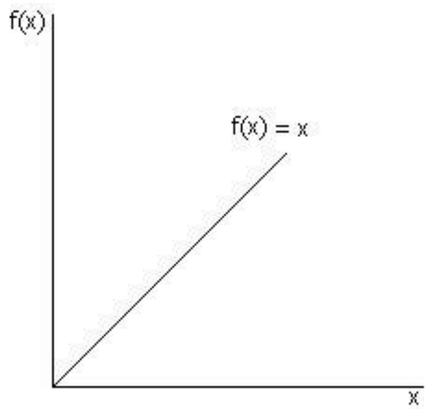


FIG 2.4 (b)



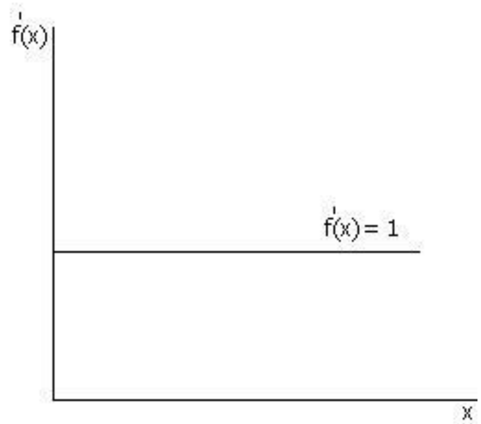
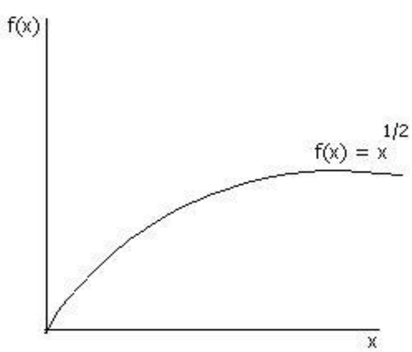
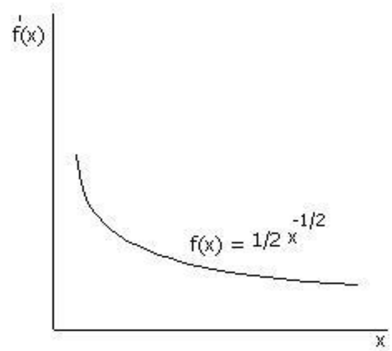


FIG 2.4 (c)





3.1. Generalised Power Function Rule

The generalised power function is symbolically written as $f(x) = g(x)^n$ where g is a differentiable function and n is any real number. The derivative of such function is

$$\boldsymbol{f'(x) = n[g(x)]^{n-1} \cdot g'(x)}$$

The above result shows, while differentiating power function, multiply the original index (n) by $g(x)$ raised to the power diminished by unity and also by the derivative $g'(x)$.

Example 2.4. Differentiate by applying power function rule to following functions

a. $f(x) = x^{-12}$ **b.** $f(x) = 3(x)^4$ **c.** $f(x) = (x^2 + 3x^4)^5$ **d.** $f(x) = \sqrt{x}$

Solution a) Given $f(x) = x^{-12} \Rightarrow f'(x) = -12x^{-12-1} = -12x^{-13}$

b) Given $f(x) = 3x^4 \Rightarrow f'(x) = 3 .4. x^{4-1} = 12x^3$.

c) Given $f(x) = (x^2 + 3x^4)^5$

Here $g(x) = (x^2 + 3x^4) \Rightarrow g'(x) = 2x + 12x^3$

$\therefore f'(x) = 5\,(x^2 + 3x^4)^{5-1}.(2x + 12x^3)$

$= 5(x^2 + 3x^4)^4.(\,2x + 12x^3)$

d) Given $f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

4. The derivative of Sum and Difference.

The derivative of sum (difference) of two functions is equal to the sum (difference) of their derivatives. i.e.

if $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$ and $(*)$

if $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$ $(**)$

The rule set out above for the case of a combination of two functions only. But, if more than two functions appear in a combination, the rules can be applied several times in succession to give the derivative. It can be noticed, however, that the sum and difference rule extends at once to give the derivative of the algebraic sum of a number of functions as the similar algebraic sum of the derivatives of the separate functions. For example,

If $f(x) = g(x) + h(x) + w(x)$, then $f'(x) = g'(x) + h'(x) + w(x)$

And if $f(x) = g(x) - h(x) - w(x)$, then $f'(x) = g'(x) - h'(x) - w(x)$

The important deductions from $(*)$ and $(**)$ concerns the behaviour of constants in the process of derivation. A constant (additive) can be regarded as the case of a function of x which does not change in value as x varies. This means that the derivative of constant is zero. Hence an additive constant disappears when the derivative is taken.[3]

To better under as to why constant if added to a function disappears on differentiating.[4] Consider two functions $f(x) = x^2$ and $f(x) = x^2 + A$, where A is a constant and plot them the figure that emerges (fig 2.5) clearly reveals that the slope (derivative) of the functions at every point is same, the only difference that is apparent is that the graph of the later function is A steps away from the graph of the former function.

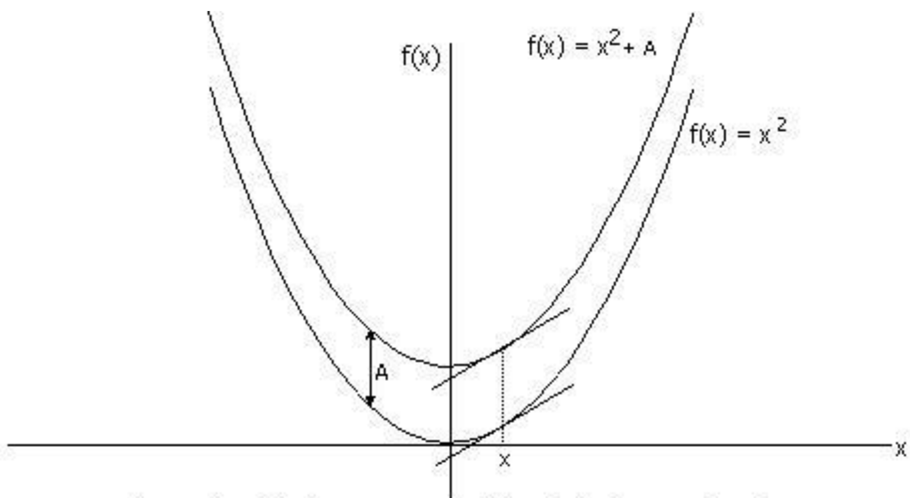


FIG 2.5 The graphs of the functions are parallel, and the functions have the same derivatives at each point.

Example 2.5. Find the derivative of the following functions.

a. $f(x) = 3x^4 + 2x^3 - x^2 + 10$
b. $f(x) = \sqrt[3]{x} + 2x - 14$
c. $f(x) = ax^2 + bx + c$; where a, b, c are constants

Solution. a) Given $f(x) = 3x^4 + 2x^3 - x^2 + 10$

[3] Multiplicative constants remain unaffected by taking the derivative of a function. If $f(x) = 2x^4$; then $f'(x) = 2\,f'(x^4) = 8x^3$

[4] This fact provides mathematical explanation of the well known economic principle that fixed costs of a firm does not affect its marginal cost.

$$f'(x) = 3f'(x^4) + 2f'(x^3) - f'(x^2)$$

$$= 3\,.4\,x^3 + 2\,.3\,x^2 - 2x$$

$$= 12\,x^3 + 6x^2 - 2x$$

b) Given $f(x) = \sqrt[3]{x} + 2x - 14$

$$f'(x) = f'(x)^{\frac{3}{2}} + 2f'(x)$$

$$= \frac{3}{2}(x)^{\frac{1}{2}} + 2$$

c) Given $f(x) = ax^2 + bx + c$

$$f'(x) = af'(x^2) + bf'(x)$$

$$= 2ax + b$$

5 product function Rule

If $f(x) = h(x).\,g(x)$, then

$f'(x) = h(x).\,g'(x) + g(x).\,h'(x)$

In words: the derivative of the product of two functions is equal to the derivative of the first function times the derivative of the second function plus the second function times the derivative of first function. The significant point to remember about the product function rule is that the derivative of the product of two functions is **not** the simple product of their separate derivatives i.e. $f'(x) \neq g'(x).\,h'(x)$. For example if $h(x) = x^2$ and $g(x) = 4x$ then $(h.\,g).(x) = (x^2).(4x) = 4x^3$ it is obvious here that $(h.\,g)'(x) = 12x^2$ is not equal to $h'(x).\,g'(x) = (2x).(4) = 8x$.

Verification by Product Rule: if $f(x) = (x)^2.(4x)$

Consider $h(x) = (x)^2$ and $g(x) = 4x$ then

By applying product rule

$$f'(x) = h(x).\,g'(x) + g(x).\,h'(x)$$

$$= x^2(4) + 4x(2x) = 4x^2 + 8x^2 = 12x^2$$

The same result was obtained when we first multiply the two functions and then differentiate the result directly. The obvious question that emerges is that why do we need product rule when we have another option of multiplying the functions and taking the derivative of the product directly. One response to the query is that multiplying first and then differentiating is applicable when the functions are given in some specific form, but the product rule is applicable even when the functions are given in general form.[5]

Example 2.6. Find the derivative of the following functions

a. $f(x) = 3x^4(2x^5 + 5x)$ **b.** $f(x) = (\sqrt{x})(\sqrt[3]{x+2})$ **c.** $f(x) = \left(x^4 + \frac{1}{x}\right)(x^3 + 2)$

solution. a) Given $f(x) = 3x^4(2x^5 + 5x)$

Let $g(x) = 3x^4$ and $h(x) = (2x^5 + 5x)$

Then $g'(x) = 12x^3$ and $h'(x) = (10x^4 + 5)$.plug in these values in product rule formula

$$f'(x) = g(x).h'(x) + h(x).g'(x)$$

$$= 3x^4(10x^4 + 5).(2x^5 + 5x).(12x^3) = 54x^8 + 75x^4$$

b) Given $f(x) = (\sqrt{x})(\sqrt[3]{x+2})$

Let $g(x) = (\sqrt{x})$ and $h(x) = \sqrt[3]{x+2})$

Differentiating $g(x)$ and $h(x)$ separately and substitute the values in the product rule formula

$g'(x) = \frac{1}{2\sqrt{x}}$ and $h(x) = (x+2)^{\frac{3}{2}} \Rightarrow h'(x) = \frac{3}{2}(x)^{\frac{1}{2}}$

$\therefore f'(x) = g(x).h'(x) + h(x).g'(x)$

$$= (\sqrt{x}).\frac{3}{2}(x)^{\frac{1}{2}} + (x+2)^{\frac{3}{2}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{3}{2}x + \frac{1}{2\sqrt{x}}(x+2)^{\frac{3}{2}}$$

c) Given $f(x) = \left(x^4 + \frac{1}{x}\right)(x^3 + 2)$

Let $g(x) = \left(x^4 + \frac{1}{x}\right)$ and $h(x) = (x^3 + 2)$

[5] For further exposition see Chiang A. C. Fundamental Methods of Mathematical Economics (4th ed.), McGraw-Hill (2005).

Differentiating $g(x)$ and $h(x)$ separately and substitute the values in the product rule formula

$g'(x) = 4x^3 - \frac{1}{x^2}$ and $h'(x) = 3x^2$

$\therefore f'(x) = g(x).h'(x) + h(x).g'(x)$

$$= \left(x^4 + \frac{1}{x}\right)(3x^2) + (x^3 + 2)(4x^3 - \frac{1}{x^2})$$

6. Derivative of the Quotient of two Functions (Quotient Rule)

If $(x) = \frac{g(x)}{h(x)}$; where $g(x)$ and $h(x)$ are both differentiable at x and $h(x) \neq 0$ then

$$f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}$$

In words: The derivative of a quotient is equal to the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Note: in the product rule formula, the two functions appear symmetrically, so that it is easy to remember. In the quotient rule however, the expression in the numerator must be in the right order. The simplest way to check whether the order of the expression in the numerator is appropriate or not, imagine that $h(x) = 1$ so that$h'(x) = 0$. If on substituting the formula reduces to $g'(x)$ then the signs are correct and if it reduces to $-g'(x)$, then the signs are wrong.

Example 2.7. Find the derivative of the following functions

a. $f(x) = \frac{x+1}{x-1}$ where $x \neq 1$ b. $f(x) = \frac{5x^6+3x^4}{2x^2-5}$ c. $f(x) = \frac{c(x)}{x}$ d. $f(x) = \frac{\sqrt{x}+1}{x}$

Solution a) Given $f(x) = \frac{x+1}{x-1}$ where $x \neq 1$

let $g(x) = x + 1$ and $h(x) = x - 1$ differentiate them individually we have.

$g'(x) = 1$ and $h'(x) = 1$ substitute the value in the formula

$$f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}$$

$$= \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

b) Given $f(x) = \frac{5x^6+3x^4}{2x^2-5}$

Let $g(x) = 5x^6 + 3x^4$ and $h(x) = 2x^2 - 5$ differentiate individually we have

$g'(x) = 30x^5 + 12x^3$ and $h'(x) = 4x$, substitute the values in the formula

$$f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}$$

$$= \frac{(2x^2-5).(30x^5+12x^3)-(5x^6+3x^4).(4x)}{(2x^2-5)^2} = \frac{40x^7-138x^5-60x^3}{(2x^2-5)^2}$$

c) Given $f(x) = \frac{c(x)}{x}$

If in the above example x is considered as the level of output and $c(x)$ as the cost of producing x units then $f(x) = \frac{c(x)}{x}$ turns out to be general average cost function.

Let $g(x) = c(x)$ and $h(x) = x$ differentiating individually we have

$g'(x) = c'(x)$ and $h'(x) = 1$ substitute the values in the formula

$$f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}$$

$$= \frac{x.c'(x)-c(x)}{x^2} = \frac{1}{x}\left[c'(x) - \frac{c(x)}{x}\right]..........(*)$$

The economic meaning of the expression $(*)$ is that for positive output levels$(x > 0)$, the marginal cost $c'(x)$ exceeds the average cost $\frac{c(x)}{x}$ if and only if the rate of change of the average cost as output increase is positive.

a) Given $f(x) = \frac{\sqrt{x}+1}{x}$

Let $g(x) = \sqrt{x} + 1$ and $h(x) = x$ differentiating g and h individually we have

$g'(x) = \frac{1}{2}\sqrt{x}$ and $h'(x) = 1$ substitute the values in the formula

$$f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}$$

$$= \frac{x\left(\frac{1}{2}\sqrt{x}\right) - (\sqrt{x} + 1)}{x^2} = \frac{\frac{1}{2}x^{\frac{3}{2}} - \sqrt{x} - 1}{x^2}$$

7. Function of a function rule Chain Rule or Composite Function Rule

If z is a function of y, and y is a function of x, that is if $z = f(y)$ and $y = g(x)$ then z is called the composite function of x and the derivative of z with respect to x is equal to the derivative of z with respect to x, times the derivative of y with respect to x.this is expressed symbolically as.

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

The main idea behind the chain rule is that the change in the value of the variable x affects the variable y, according to the function $y = g(x)$, and a change in the variable y, in turn, affects the variable z according to the function $z = f(y)$.

Example 2.8. Differentiate the following (Using Chain Rule)

a. $z = y^{10}$ Where $y = 2 + 3x$ **b.** $y = (3x^2 + 7)^{10}$ **c.** $y = x - x^6$;where $x = \frac{1}{z} + 1$

Solution.

a) Given $z = y^{10}$ Where $y = 2 + 3x$

Differentiate z with respect to y and y with respect to x

$$\frac{dz}{dy} = 10y^9 \text{ and } \frac{dy}{dx} = 3$$

Substitute the values in the formula

$$\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx} = 10y^9.3 = 30y^9 = 30(2 + 3x)^9$$

b) Given $y = (3x^2 + 7)^{10}$

let $y = u^{10}$ where $u = 3x^2 + 7$

Differentiating y with respect to u and u with respect to x

$$\frac{dy}{du} = 10u^9 \text{ and } \frac{du}{dx} = 6x$$

Substitute the values in the formula

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 10u^9.6x = 60x.u^9 = 60x(3x^2 + 7)^9$$

c) Given $y = x - x^6$; where $x = \frac{1}{z} + 1$

Differentiate y with respect to x and x with respect to z

$\frac{dy}{dx} = 1 - 6x^5$ and $\frac{dx}{dz} = \frac{-1}{z^2}$

Substitute the values in the formula

$\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = (1 - \ 6x^5)\left(\frac{-1}{x^2}\right) = 1 - 6(\frac{1}{z} + 1)^5.(\frac{-1}{z^2})$

Example 2.9. Given a profit function of a firm $\pi = f(y)$, where output y is a function of labour input L, or $y = g(L)$,find $\frac{d\pi}{dL}$ By using Chain rule.

Solution. Given $\pi = f(y)$ and $y = g(L)$

Differentiating π with respect to y and y with respect to L

π' Or $\frac{d\pi}{dy} = f'(y)$ and y' Or $\frac{dy}{dL} = g'(L)$

Substitute the values in the formula

$\frac{d\pi}{dL} = \frac{d\pi}{dy}.\frac{dy}{dL}$ Or $f'(g(L).\ g'(L).$ [6]

8. Inverse Function Rule.

If $= f(x)$, an inverse function $x = g(y)$,that is $g(y) = \ f^{-1}(y)$ exists if each value of y yields one and only one value of x.if the inverse function exists, the inverse function rule states that the derivative of the inverse function is the reciprocal of the derivative of the original function that is

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Alternatively we can write

$$f^{-1}(y) = \frac{1}{f'(x)}$$

Example 2.10. Find the derivative for the inverse of the following functions:

[6] The Leibniz's notation of remembering chain rule although easy suffers from the defect of not specifying where the derivatives ($\frac{d\pi}{dy}$, $\frac{dy}{dL}$) are actually calculated second way of writing the chain rule i.e. $f'(g(L).\ g'(L)$ makes it obvious that f' is calculated at $g(L)$ where as g' is calculated at L.

a. $y = mx$ **b.** $q = L^{1/2}$ **c.** $L = q^2$ **d.** $Q = 30 - 3P$

Solution. a) Given $y = mx$

Differentiating with respect to x we have

$$\frac{dy}{dx} = m$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{m}$$

b) Given $q = L^{1/2}$

Differentiating with respect to L we have

$$\frac{dq}{dL} = \frac{1}{2\sqrt{L}}$$

$$\therefore \quad \frac{dL}{dq} = \frac{1}{\frac{dq}{dL}} = \frac{1}{\frac{1}{2\sqrt{L}}} = 2\sqrt{L}$$

c) Given $L = q^2$

Differentiating with respect to q we have

$$\frac{dL}{dq} = 2q$$

$$\therefore \quad \frac{dq}{dL} = \frac{1}{\frac{dL}{dq}} = \frac{1}{2q}$$

D) Given $Q = 30 - 3P$

Differentiating with respect to P we have

$$\frac{dQ}{dp} = -3$$

$$\therefore \quad \frac{dp}{dQ} = \frac{-1}{3}$$

1.1. Higher Order Derivatives : Convexity and Concavity of a Function

Up to this point we were dealing with the first order derivatives $f'(x)$ or $\frac{dy}{dx}$ .since the first derivative of a function is also a function of x, it too should be differentiable with respect to x ,provided the conditions of differentiability are first satisfied. The method of obtaining the second and even higher order derivatives of a function introduces nothing new. Once the first order derivative is obtained by the suitable method already discussed, the second order derivative is obtained by further use of the rules, this time applied to the first derivative is considered as a function of x.the second order derivative of a function $f(x)$ denoted by

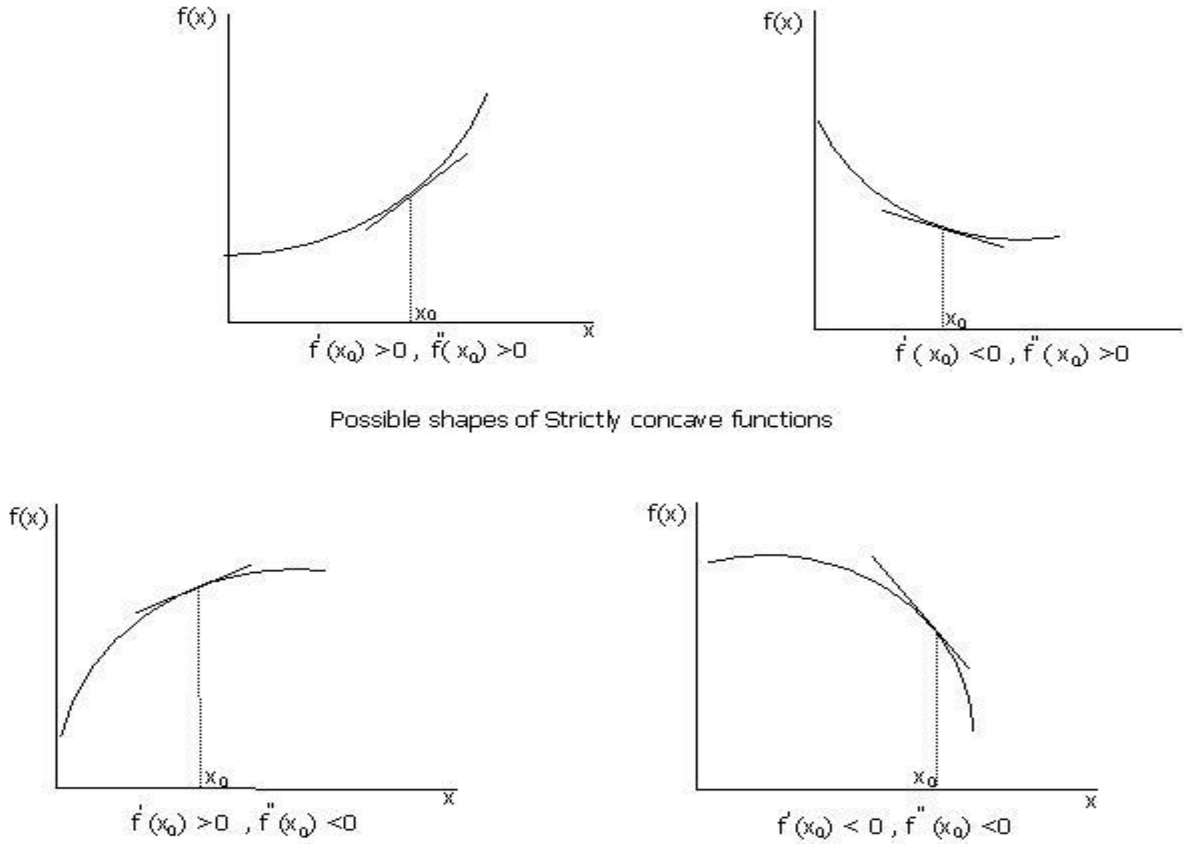
$$f''(x) \quad \text{or} \quad \frac{d(f'(x))}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2}$$

In economics we obtain many useful results by focussing on the first and second derivative of a function. The sign, positive or negative in particular, of a second derivative of a function leads us to an important and simple method of determining the convexity or concavity of a function.

- A twice differentiable function $f(x)$ is strictly convex at $x = x_0$ if $f''(x_0) > 0$. This means that the function $f(x)$ changes at an increasing rate as x increases through the value x_0.
- A twice differentiable function $f(x)$ is strictly concave at $x = x_0$ if $f''(x_0) < 0$. This means that the function $f(x)$ changes at a decreasing rate as x increases through the value x_0.

A function whose second derivative is sometimes positive and sometimes negative is neither convex nor concave everywhere. However, we can, sometimes find intervals over which the function is either convex or concave. Some possible shapes of strictly convex and concave functions are shown in fig. 2.6

FIG 2.6 possible shapes of the strictly convex functions



Example 2.11. check the convexity or concavity of the following.

a. $f(x) = 10 - x^2$ **b.** $f(x) = x^2$ **c.** $f(x) = -\frac{2}{3}(x)^3 + 10x^2 + 5x$; where $x \geq 0$

Solution.

a) Here $f(x) = 10 - x^2$

Differentiating with respect to x

$f'(x) = -2x$

Differentiating again with respect to x we have

$f''(x) = -2 < 0$

Since the second derivative is less than zero, it confirms that the given function is strictly concave.[7]

b) Given $f(x) = x^2$

Differentiating with respect to x

$f'(x) = 2x$

Differentiating again with respect to x we have

$f''(x) = 2 > 0$

Since the second derivative is greater than zero, it confirms that the given function is strictly convex.

c) Given $f(x) = -\frac{2}{3}(x)^3 + 10x^2 + 5x$

Differentiating with respect to x

$f'(x) = -2x^2 + 20x + 5$

Differentiating again with respect to x

$f''(x) = -4x + 20$

Since x can take values greater than or equal to zero. It follows that the function is convex on the interval [0, 5) and concave on the interval (5,∞).That is

$f''(x) = -4x + 20 > 0$ When $x < 5$ (convex)

$f''(x) = -4x + 20 < 0$ When $x > 5$ (concave)

[7] The precise geometric classification of a strictly concave function is that if we choose any two points on its curve and join them by a straight line, the line segment will lie completely below the curve except at the end points. For strict convex functions, the line segment will lie entirely above the curve.