Exercise: Solve the partial differential equation $p_2 + p_3 = 1 + p_1$,

where $p_1 = \frac{\partial z}{\partial x_1}$, $p_2 = \frac{\partial z}{\partial x_2}$, $p_3 = \frac{\partial z}{\partial x_3}$

Sol: The given equation can be written as

$$p_1 - p_2 - p_3 = -1$$

which is linear PDE

The Auxiliary equations are

$$\frac{dx_1}{1} = \frac{dx_2}{-1} = \frac{dx_3}{-1} = \frac{dz}{-1} \qquad \dots (1.60)$$

Taking the first two fractions of (1.60) we get

$$\frac{dx_1}{1} = \frac{dx_2}{-1}$$

Integrating we get

$\Rightarrow$ $x_1 = -x_2 + c_1$

$\Rightarrow$ $x_1 + x_2 = c_1$

Taking the 2nd and 3rd fractions of (1.60) we get

$$\frac{dx_2}{-1} = \frac{dx_3}{-1}$$

Integrating we get

$\Rightarrow$ $-x_2 = -x_3 + c_2$

$\Rightarrow$ $x_3 - x_2 = c_2$

Taking the 3rd and 4th fractions of (1.60) we get

$$\frac{dx_3}{-1} = \frac{dz}{-1}$$

Integrating we get

$\Rightarrow$ $-x_3 = -z + c_3$

$\Rightarrow$ $z - x_3 = c_3$

The general solution is given by $f(c_1, c_2, c_3)$

or $f(x_1 + x_2, x_3 - x_2, \ z - x_3)$

Exercise: Solve $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = az + \frac{xy}{t}$

Solution: The given equation is a PDE in three independent variables $x,\ y,\ t$,

Therefore the auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az+\frac{xy}{t}} \qquad \dots (1.61)$$

Taking the 1st and 2nd fraction of (1.61) we get

$$\Rightarrow \qquad \frac{dx}{x} = \frac{dy}{y}$$

Integrating

$$\log x = \log y + \log c_1$$

$$\Rightarrow \qquad \frac{x}{y} = c_1$$

Taking the 2nd and 3rd fraction of (1.61) we get

$$\Rightarrow \qquad \frac{dy}{y} = \frac{dt}{t}$$

Integrating

$$\log y = \log t + \log c_2$$

$$\Rightarrow \qquad \frac{y}{t} = c_2$$

Taking the 3rd and 4th fraction of (1.61) we get

$$\Rightarrow \qquad \frac{dt}{t} = \frac{dz}{az+\frac{xy}{t}}$$

$$\Rightarrow \qquad \frac{dt}{t} = \frac{dz}{az+xc_2}$$

Integrating

$$\log t = \frac{1}{a}\log(az+xc_2) + \log c_3$$

$$\Rightarrow \qquad \log t = \log(az+xc_2)^{\frac{1}{a}} + \log c_3$$

$$\Rightarrow \qquad \frac{t}{(az+xc_2)^{\frac{1}{a}}} = c_3$$

$$\Rightarrow \qquad \frac{t}{\left(az+x\frac{y}{t}\right)^{\frac{1}{a}}} = c_3$$

The general solution is given by $f(c_1, c_2, c_3) = 0$

or $$f\left[\frac{x}{y},\frac{y}{t},\frac{t}{\left(az+x\frac{y}{t}\right)^{\frac{1}{a}}}\right] = 0$$

Exercise: Solve $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=xyz$

Solution: The given equation is a PDE in three independent variables $x,\ y,\ z$,

Therefore the auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{xyz} \quad \ldots(1.62)$$

Taking the 1st and 2nd fraction of (1.62) we get

$\Rightarrow$ $$\frac{dx}{x}=\frac{dy}{y}$$

Integrating

$$\log x=\log y+\log c_1$$

$\Rightarrow$ $$\frac{x}{y}=c_1$$

Taking the 2nd and 3rd fraction of (1.62) we get

$\Rightarrow$ $$\frac{dy}{y}=\frac{dz}{z}$$

Integrating

$$\log y=\log z+\log c_2$$

$\Rightarrow$ $$\frac{y}{z}=c_2$$

Using $(yz,\ zx,\ xy)$ as multipliers then each fraction (1.62) is equal to

$$\frac{yzdx+zxdy+xydz}{xyz+xyz+xyz}$$

Equating this expression with the 4th fraction of (1.62) we get

$$\frac{yzdx+zxdy+xydz}{3xyz}=\frac{du}{xyz}$$

$\Rightarrow$ $$d(xyz)=3du$$

Integrating we get

$$xyz=3u+c_3$$

$\Rightarrow$ $$xyz-3u=c_3$$

The general solution is given by $f(c_1,\ c_2,\ c_3) = 0$

or $$f\left(\frac{x}{y},\ \frac{y}{z},\ xyz - 3u\right) = 0$$

Integral surface passing through a given a curve

Consider the first order linear PDE $Pp + Qq = R$

We know that the auxiliary system associated with the given PDE is given by

$$\frac{dx}{Q} = \frac{dy}{P} = \frac{dz}{R}$$

Let $u(x, y, z) = c_1$ and $\mathrm{v}(x, y, z) = c_2$ represent the integral surface of the above system.

Suppose that $[x(t),\ y(t),\ z(t)]$ be the parametric form of the curve passing through the above integral surface

i.e., $u[x(t),\ y(t),\ z(t)] = 0$

and $\mathrm{v}[x(t),\ y(t),\ z(t)] = 0$

The general integral of the given PDE is $f(u, v) = 0$, subject to the condition that $f(c_1, c_2\ ) = 0$

Exercise: Find the equation of the integral surface of the PDE

$$2y(z-3)p + (2x - z)q = y(2x - 3)$$

Sol: The auxiliary system is given by

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \qquad \dots (1.62a)$$

Taking the 1st and 3rd fraction of (1.62a), we get

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$\Rightarrow \qquad \frac{dx}{2(z-3)} = \frac{dz}{2x-3}$$

$$\Rightarrow \qquad (2x-3)dx = (2z-6)dz$$

Integrating we get

$$x^2 - 3x = z^2 - 6z + c_1$$

or $$x^2 - 3x - z^2 + 6z = c_1 \qquad \dots (1.63)$$

Using $(0,\ y, -1)$ as multipliers each fraction of (1.62a) is equal to

$$\frac{ydy - dz}{2xy - yz - 2yx + 3y} = \frac{ydy - dz}{-yz + 3y} = \frac{ydy - dz}{y(3 - z)}$$

Equating this expression with 1st fraction of (1.62a) we get

$$\frac{dx}{2y(z-3)} = \frac{ydy-dz}{y(3-z)}$$

$\Rightarrow$ $$\frac{dx}{2y(z-3)} = \frac{ydy-dz}{y(3-z)}$$

$\Rightarrow$ $$\frac{dx}{2} = \frac{ydy-dz}{-1}$$

$\Rightarrow$ $$dx + 2ydy - 2dz = 0$$

Integrating we get

$$x + y^2 - 2z = c_2 \qquad \ldots(1.64)$$

Now the given curve is $x^2 + y^2 = 2x, \quad z = 0,$

This equation can also be written as $(x-1)^2 + (y-0)^2 = 1$ which is circle with centre $(1,0)$ and radius 1. The corresponding parametric equation is

$$x - 1 = \cos\theta, \quad y - 0 = \sin\theta, \quad z = 0.$$

or $$x = 1 + \cos\theta, \quad y = \sin\theta, \quad z = 0.$$

By the given condition, the integral surface passes through the above circle. Therefore from (1.64) we get

$$1 + \cos\theta + \sin^2\theta - 2(0) = c_2$$

$\Rightarrow$ $$1 + \cos\theta + 1 - \cos^2\theta = c_2$$

$\Rightarrow$ $$2 + \cos\theta - \cos^2\theta = c_2 \qquad \ldots(1.65)$$

Also from (1.63), we get

$$(1 + \cos\theta)^2 - 3(1 + \cos\theta) - 0^2 + 6(0) = c_1$$

$\Rightarrow$ $$1 + 2\cos\theta + \cos^2\theta - 3 - 3\cos\theta = c_1$$

$\Rightarrow$ $$\cos^2\theta - 2 - \cos\theta = c_1$$

$\Rightarrow$ $$-(2 + \cos\theta - \cos^2\theta) = c_1 \qquad \ldots(1.66)$$

From (1.65) and (1.66), we get

$$c_1 = -c_2$$

or $$c_1 + c_2 = 0$$

or $$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$$

or $$x^2 + y^2 - z^2 - 2x + 4z = 0$$

which is required surface.

Integral surface orthogonal to given surface:

Consider the linear partial differential equation

$$Pp + Qq = R \qquad \ldots(1.67)$$

Let $$f(x, y, z) = c \qquad \ldots(1.68)$$

be the integral surface of (1.67), also for any surface

$$z = g(x, y) \qquad \ldots(1.69)$$

Let $p(x, y, z)$ be any point on the line such that (1.68) and (1.69) are orthogonal at p.

We have the direction ratios respectively for (1.68) and (1.69) as

$$< f_x, \ f_y, \ f_z > \quad \text{and} \quad < p, \ q, -1 >$$

Now the condition for the orthogonality suggests that

$$f_x p + f_y q + f_z(-1) = 0$$

or $$f_x p + f_y q = f_z$$

which is of the form $Pq + Qq = R$

where $P = f_x, \ \ Q = f_y, \quad R = f_z,$

Exercises: Find the surface which is orthogonal to one parameter system

$z = cxy(x^2 + y^2)$ and passes through the hyperbola $x^2 - y^2 = a^2, \quad z = 0.$

Solution: The given one parameter system is $\frac{xy(x^2+y^2)}{z} = \frac{1}{c}$

Let $$f(x, y, z) = \frac{xy(x^2+y^2)}{z}$$

Now, $$\text{P}= f_x = \frac{y(x^2+y^2)+2x^2y}{z}$$

$$Q = f_y = \frac{x(x^2+y^2)+2xy^2}{z}$$

and $$R = f_z = \frac{-xy(x^2+y^2)}{z^2}$$

now auxiliary system of equations are

$$\frac{dx}{\frac{y(x^2+y^2)+2x^2y}{z}} = \frac{dy}{\frac{x(x^2+y^2)+2xy^2}{z}} = \frac{dz}{\frac{-xy(x^2+y^2)}{z^2}}$$

or $$\frac{dx}{y(x^2+y^2)+2x^2y} = \frac{dy}{x(x^2+y^2)+2xy^2} = \frac{zdz}{-xy(x^2+y^2)} \qquad \ldots(1.70)$$

Using multipliers $(x, y, 1)$, each ratio of (1.70) is equal to

$$\frac{xdx+ydy+zdz}{3x^3y+xy^3+x^3y+3xy^3-x^3y-xy^3} = \frac{xdx+ydy+zdz}{3xy(x^2+y^2)}$$

Equating this with 3rd term of (1.70), we get

$$\frac{xdx+ydy+zdz}{3xy(x^2+y^2)} = \frac{zdz}{-xy(x^2+y^2)}$$

$\Rightarrow$ $$xdx + ydy + zdz = -3zdz$$

$\Rightarrow$ $$xdx + ydy + 4zdz = 0$$

Integrating

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{4z^2}{2} = \frac{c_1}{2}$$

$\Rightarrow$ $$x^2 + y^2 + 4z^2 = c_1 \qquad \ldots (1.71)$$

Using multipliers $(x, y, 0)$ and $(x, -y,\ 0)$ and equating the two fractions we get

$$\frac{xdx+ydy}{3x^3y+xy^3+x^3y+3xy^3} = \frac{xdx-ydy}{3x^3y+xy^3-x^3y-3xy^3}$$

$\Rightarrow$ $$\frac{xdx+ydy}{4x^3y+4xy^3} = \frac{xdx-ydy}{2x^3y-2xy^3}$$

$\Rightarrow$ $$\frac{xdx+ydy}{x^2+y^2} = \frac{2(xdx-ydy)}{x^2-y^2}$$

Integrating, we get

$$\log(x^2 + y^2) = 2\log(x^2 - y^2) + \log c_2$$

$\Rightarrow$ $$\frac{x^2+y^2}{(x^2-y^2)^2} = c_2 \qquad \ldots(1.72)$$

Now parametric systems of hyperbola is

$$x = a\sec\theta\,, \quad x = a\tan\,\theta\,, \quad z = 0$$

$\therefore$ from (1.72),

$$c_1 = a^2\ \sec^2\theta + a^2\tan^2\theta$$

$\Rightarrow$ $$c_1 = a^2\,(\sec^2\theta + \tan^2\theta)$$

And from (1.72)

$$c_2 = \frac{a^2 \sec^2\theta + a^2\tan^2\theta}{(a^2 \sec^2\theta - a^2\tan^2\theta)^2}$$

$$\Rightarrow \quad c_2 = \frac{\sec^2\theta + \tan^2\theta}{a^2(\sec^2\theta - \tan^2\theta)^2}$$

$$\Rightarrow \quad c_2 = \frac{\sec^2\theta + \tan^2\theta}{a^2(1)^2}$$

$$\Rightarrow \quad c_2 = \frac{\frac{c_1}{a^2}}{a^2(1)^2}$$

$$\Rightarrow \quad c_2 = \frac{c_1}{a^4}$$

$$\Rightarrow \quad a^4 c_2 = c_1$$

$$\Rightarrow \quad c_1 = a^4 c_2$$

∴ The required surface orthogonal to the given system is

$$\frac{(x^2-y^2)^2(x^2+y^2+4z^2)}{x^2+y^2} = a^4$$

Standard forms of Partial differential equations:

Standard form I: If the given partial differential equations contains p and q

i.e., $f(p,q) = 0$

Clearly in this equation the variables x, y, z are absent. The solution of such type of equation is

$$z = ax + by + c$$

or

$$z = ax + \phi(a)y + c$$

Such type of equations does not posses the singular integral, since

$$\frac{\partial z}{\partial a} = x + \phi'(a)y = 0$$

And

$$\frac{\partial z}{\partial c} = 1 \neq 0$$

Exercise: $p^2 + q^2 = 1$ … (1.73)

Solution: It is of the standard form as it does contains p and q only

Let $f(p,q) = p^2 + q^2 - 1 = 0$

Therefore the solution is

$$z = ax + by + c$$

Where a and b satisfy (1.73) so that $a^2 + b^2 = 1$

$\Rightarrow$ $$b = \sqrt{1 - a^2}$$

$\therefore$ The complete solution is $z = ax + \sqrt{1 - a^2}\ y + c$

Exercise: Solve the partial differential equaiton $p^2 - q^2 = 4$

Exercise: Solve the partial differential equaiton $q = 3p^2$ …(1.74)

Solution: It is a standad form of PDE of category- I

Let $$f(p, q) = q - 3p^2 = 0$$

Therefore the solution is

$$z = ax + by + c$$

where a and b satisfy (1.74) so that $a - 3b^2 = 0$

$\Rightarrow$ $$b = \sqrt{\frac{a}{3}}$$

$\therefore$ The complete solution is

$$z = ax + \sqrt{\frac{a}{3}}\ y + c$$

Exercise: Solve the partial differential equaiton $p + q = pq$

Exercise: Solve the partial differential equaiton $x^2p^2 + y^2q^2 = z^2$ …(1.75)

Sol: Given that $$\left(\frac{x}{z}\, p\right)^2 + \left(\frac{y}{z}\, q\right)^2 = 1$$

or $$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

Put $$x = e^X, \quad y = e^Y, \quad z = e^Z$$

or $$\log x = X, \quad \log y = Y, \quad \log z = Z$$

$\Rightarrow$ $$\frac{1}{x} = \frac{\partial X}{\partial x}, \quad \frac{1}{y} = \frac{\partial Y}{\partial y}, \quad \frac{1}{z} = \frac{\partial Z}{\partial z}$$

Now $$\frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial x}\frac{\partial x}{\partial X} = x\frac{\partial Z}{\partial x} = x\frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = \frac{x}{z}\frac{\partial z}{\partial x}$$

$$\frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial y}\frac{\partial y}{\partial Y} = y\frac{\partial Z}{\partial y} = y\frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} = \frac{y}{z}\frac{\partial z}{\partial y}$$

Using these in (1.75), we get

$$\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

or $$P^2 + Q^2 = 1$$

It is a PDE of standard form $f(p, q) = 0$,

$\therefore$ Its solution is

$$z = aX + bY + c$$

where a and b are connected by $a^2 + b^2 = 1$

$\Rightarrow$ $$b = \sqrt{1 - a^2}$$

Standard from II

The PDE of the form $f(p, q, z) = 0$, where x and y are explicitly absent is called the standard form II.

The trivial solution of such PDE's is given by

$$z = f(x + ay) = f(X)$$

where $X = x + ay$.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = a\frac{\partial z}{\partial Y}$$

Substituting the values of p and q in the given equation we get an ordinary differential equation with x as independent variable and Xas dependent variable.

Exercise: Solve the partial differential equaiton $p^2 = zq$ …(1.76)

Sol: The given equation of the standard form II.

Therefore, let $f(p, q, z) = p^2 - zq = 0$

Put $X = x + ay$,

So that $p = \frac{\partial z}{\partial X}$ and $q = a\frac{\partial z}{\partial X}$

Substituting values of p and q in (1.76) we get

$$\left(\frac{dz}{dX}\right)^2 = za\frac{dz}{dX}$$

$\Rightarrow$ $$\frac{dz}{dX} = za$$

$\Rightarrow \qquad \frac{dz}{z} = adX$

Integrating

$$\log z = aX + c_1$$

$\Rightarrow \qquad z = e^{aX+c_1}$

$\Rightarrow \qquad z = c_2 e^{aX}$

$\Rightarrow \qquad z = c_2 e^{a(x+ay)}$

This is required solution.

Exercise: Solve the partial differential equaiton $9(p^2 z + q^2) = 4$

Exercise: Solve the partial differential equaiton $p^3 + q^3 = 27z$

Exercise: Solve the partial differential equaiton $pz = 1 + q$ …(1.77)

Solution: The given equation of the standard form II.

Therefore let $\qquad f(p, q, z) = pz - 1 - q = 0$

Put $\qquad X = x + ay,$

so that $\qquad p = \frac{\partial z}{\partial X}$ and $q = a\frac{\partial z}{\partial X}$

Substituting values of p and q in (1.77), we get

$$\left(\frac{dz}{dX}\right) z - 1 - a\frac{dz}{dX} = 0$$

$\Rightarrow \qquad \frac{dz}{dX} = \frac{1}{z-a}$

$\Rightarrow \qquad (z-a)dz = dX$

Integrating

$$\frac{z^2}{2} - az = X + c$$

$\Rightarrow \qquad z^2 - 2az = 2X + 2c$

$\Rightarrow \qquad z(z - 2az) = 2(x + ay) + 2c$

$\Rightarrow \qquad z(z - 2az) = 2(x + ay) + b$

This is required solution.

Exercise: $P(1 + p^2) = q(z - a)$

Standard from III

The PDE of the form $f(p,x) = F(q,y)$ is called the standard form III.

Equating these equations to the constant a, and finding the values of p and q, and using these values of p and q in

$$dz = pdx + qdy$$

Hence the solution of the above equation represents the solution of the given equation.

Exercise: Solve the partial differential equaiton $\sqrt{P} + \sqrt{q} = 2x$

Solution: We can write this equation as

$$\sqrt{P} - 2x = -\sqrt{q}$$

Therefore, let $\sqrt{P} - 2x = a$ and $-\sqrt{q} = a\ (say)$

Then $q = a^2$

so that $\sqrt{P} = a + 2x \Rightarrow p = (a + 2x)^2$

Using the values of p and q in

$$dz = pdx + qdy$$

or

$$dz = (a + 2x)^2 dx + a^2 dy$$

Integrating, we get

$$z = \frac{(a+2x)^3}{3(2)} + a^2 x + b$$

Exercise: $p^2 + q^2 = x + y$

Exercise: Solve the partial differential equaiton $q = xyp^2$

Solution: The equation can be written as

$$xp^2 = \frac{q}{y}$$

which is of standard form III,

Therefore let $xp^2 = a$ and $\frac{q}{y} = a$

$$p = \sqrt{\frac{a}{x}} \quad \text{and} \quad q = ay$$

Using the values of p and q in

$$dz = pdx + qdy$$

$$dz = \sqrt{\frac{a}{x}}dx + aydy$$

Integrating, we get

$$z = 2\sqrt{a}(x)^{\frac{1}{2}} + a\frac{y^2}{2} + b$$

$$2z = 4\sqrt{a}(x)^{\frac{1}{2}} + ay^2 + 2b$$

$$2z - ay^2 - 2b = 4\sqrt{a}(x)^{\frac{1}{2}}$$

Squaring on the both sides we get

$$16ax - (2z - ay^2 - 2b)^2 = 0$$

which is required complete integral.

Exercise: Solve the partial differential equaiton $pq = xy$

Exercise: Solve the partial differential equaiton

$$yp - 2xy = \log q \qquad \ldots(1.78)$$

Solution: The equation can be written as

$$p - 2x = \frac{\log q}{y}$$

which is of standard form III,

Therefore let $p - 2x = a$ and $\frac{\log q}{y} = a$

Implies $p = a + 2x$ and $q = e^{ay}$

Using the values of p and q in the following equation

$$dz = pdx + qdy$$

or $$dz = (a + 2x)dx + e^{ay}dy$$

Integrating we get

$$z = ax + x^2 + \frac{e^{ax}}{a} + b$$

Which is required complete integral.

Standard Form IV (Clairaut's form)

Since we know that the complete primitive of $y = px + f(p)$ is $y = cx + f(c)$

Similarly the complete integral of PDE $z = px + qy + f(p,q)$ is $z = ax + by + f(a,b)$

Exercise: $z = px + qy + p^2 + q^2$

Sol: The equation is of Clairaut's form therefore its complete solution is given by

$$z = ax + by + a^2 + b^2 \qquad \dots (1.79)$$

Differtiating (1.79) with respect to a and b we get

$$\frac{\partial z}{\partial a} = x + 2a \qquad \text{and} \qquad \frac{\partial z}{\partial b} = y + 2b$$

Now $\qquad \frac{\partial z}{\partial a} = 0 \;\Rightarrow\; a = -\frac{x}{2}$

and $\qquad \frac{\partial z}{\partial b} = 0 \;\Rightarrow\; b = -\frac{y}{2}$

Using these values of a and b in (1.79) we get

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{x^2}{4}$$

Exercise: $z = px + qy + c\sqrt{1 + p^2 + q^2}$

Solution: The equation is of Clairaut's form therefore its complete solution is given by

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \qquad \dots (1.80)$$

Diff. this with respect to a and b we get

$$\frac{\partial z}{\partial a} = x + \frac{ca}{\sqrt{1+a^2+b^2}}$$

Let $\qquad f = z - ax - by - c\sqrt{1 + a^2 + b^2} \qquad \dots (1.81)$

Differentiating w. r. t., a and b

$$\frac{\partial f}{\partial a} = -x - \frac{ca}{\sqrt{1+a^2+b^2}} \qquad \text{and} \qquad \frac{\partial f}{\partial b} = -y - \frac{cb}{\sqrt{1+a^2+b^2}}$$

Equating to zero these partial derivatives we have

$$-x - \frac{ca}{\sqrt{1+a^2+b^2}} = 0 \qquad \text{and} \qquad -y - \frac{cb}{\sqrt{1+a^2+b^2}} = 0$$

$$\Rightarrow \qquad -x - \frac{ca}{\sqrt{1+a^2+b^2}} = 0 \qquad \text{and} \qquad -y - \frac{cb}{\sqrt{1+a^2+b^2}} = 0$$

$$\Rightarrow \qquad -x = \frac{ca}{\sqrt{1+a^2+b^2}} = 0 \qquad \text{and} \qquad -y = \frac{cb}{\sqrt{1+a^2+b^2}}$$

$\Rightarrow \qquad \frac{-x}{ca} = \frac{1}{\sqrt{1+a^2+b^2}}$ and $\frac{-y}{cb} = \frac{1}{\sqrt{1+a^2+b^2}}$

$\Rightarrow \qquad \frac{x^2}{c^2a^2} = \frac{1}{1+a^2+b^2}$ and $\frac{y^2}{c^2b^2} = \frac{1}{1+a^2+b^2}$

$$\Rightarrow \qquad \frac{c^2a^2}{x^2} = 1 + a^2 + b^2 \qquad \text{… (1.82)}$$

$$\text{and} \qquad \frac{c^2b^2}{y^2} = 1 + a^2 + b^2 \qquad \text{…(1.83)}$$

From these two equations, we get

$$\frac{c^2a^2}{x^2} = \frac{c^2b^2}{y^2}$$

$$\Rightarrow \qquad a = \frac{bx}{y} \qquad \text{…(1.84)}$$

From (1.83), we get

$$\frac{c^2b^2x^2}{x^2y^2} = 1 + \frac{b^2x^2}{y^2} + b^2$$

$$\Rightarrow \qquad \frac{c^2b^2}{y^2} = \frac{y^2+b^2x^2+b^2y^2}{y^2}$$

$$\Rightarrow \qquad c^2b^2 = y^2 + b^2x^2 + b^2y^2$$

$$\Rightarrow \qquad (c^2 - x^2 - y^2)b^2 = y^2$$

$$\Rightarrow \qquad b = \frac{y}{\sqrt{c^2-x^2-y^2}}$$

From (1.84) we get

$$a = \frac{x}{\sqrt{c^2-x^2-y^2}}$$

using these values of a and b in (1.84)

$$z = \frac{x^2}{\sqrt{c^2-x^2-y^2}} + \frac{y^2}{\sqrt{c^2-x^2-y^2}} + c\sqrt{1 + \frac{x^2}{c^2-x^2-y^2} + \frac{y^2}{c^2-x^2-y^2}}$$

$$\Rightarrow \qquad z = \frac{x^2}{\sqrt{c^2-x^2-y^2}} + \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$\Rightarrow \qquad z^2(c^2 - x^2 - y^2) = (x^2 + y^2 + c^2)^2$$

which is required solution.