The main objective of this unit is to discuss the concept of an infinite series their convergence , conditional, absolute and the process of their rearrangements. We will be discussing certain tests for convergence of sequence and series namely Abel's Test, Drichlet's Test and some important theorem like Carleman's theorem, Dirchlet's theorem and many others. In the end of this unit we will consider the sequence and series of functions and we prove their convergence and uniform convergence by important tests.

Definition Infinite series 1.1. We are already familiar with arithmetic and geometric series. In an arithmetic series, each term after the first term is formed by adding a fixed number to the proceeding term and in a geometric series, each term after the first is formed by multiplying the proceedings term by a fixed number. A series can be made by other ways also. For example, the series 1+4+9+16+25+36 is formed by the squares of the first six natural numbers. A series is the sum of the terms of sequence.

Thus if $u_1, u_2 \ldots\ldots$is a sequence, then the sum $u_1 + u_2 \ldots\ldots$of all terms is called an infinite series and is denoted by $\sum_{i=1}^{\infty} u_n$ or simply by $\sum u_n$.

If we denoted S_n by $u_1 + u_2 + \ldots\ldots\ldots\ldots u_n$.

That is ,

$$S_n \;=\; u_1 + u_2 + \cdots \ldots \ldots . u_n$$

Then the sequence $< S_n >$ is called a sequence of partial sums of series and the partial sums,

$$S_1 = u_1 , \;\; S_2 = u_1 + u_2 ,\ldots\ldots\ldots . \;\; S_n$$

$$and \quad S_n \;=\; u_1 + u_2 + \cdots \ldots \ldots . u_n$$

and so on may be required as approximation to infinite series $\sum u_n$.

The series $\sum u_n$ is convergent if the sequence of partial sums are convergent,

That is if $\lim_{n\to\infty} S_n$ exits,

Then, $\sum u_n$ is convergent and we write $\lim_{n\to\infty} S_n = \sum u_n$.

Carleman's Theorem 1.1 .

Suppose $\sum a_n$ be convergent series of positive terms , then

$\sum_{n=1}^{\infty}(a_1 a_2 \ldots \ldots a_n)^{\frac{1}{n}}$ is convergent and $\sum u_n. \leq e \sum a_n$.

where $u_n = (a_1 a_2 \ldots \ldots \ldots \ldots a_n)^{\frac{1}{n}}$.

Proof: We define c_n by

$$c_1 c_2 \ldots \ldots c_n = (n+1)^n \qquad (1)$$

$$c_1 c_2 \ldots \ldots . c_{n-1} = (n)^{n-1} \qquad (2)$$

Dividing (i) and (ii), we get

$$c_n = \frac{(n+1)^n}{(n)^{n-1}}$$

This implies that $\frac{c_n}{n} = (\frac{n+1}{n})^n$ (3)

Now

$$\sum(a_1 a_2 \ldots \ldots \ldots \ldots a_n)^{\frac{1}{n}} = \sum(a_1 a_2 \ldots \ldots \ldots \ldots a_n)^{\frac{1}{n}} . 1$$

$$= \sum(a_1 a_2 \ldots \ldots a_n)^{\frac{1}{n}} . \frac{(c_1 c_2 \ldots \ldots c_n)^{\frac{1}{n}}}{n+1}$$

This implies $\sum(a_1 a_2 \ldots \ldots \ldots \ldots a_n)^{\frac{1}{n}} = \frac{\sum(c_1 c_2 \ldots \ldots \ldots c_n)^{\frac{1}{n}}}{n+1}$ (4)

By Arithmetic-Geometric Mean Inequality, we get from (4)

$$\sum(a_1 a_2 \ldots\ldots\ldots.. a_n)^{\frac{1}{n}} \leq \sum \frac{a_1 c_1 + a_2 c_2 + \cdots \ldots\ldots\ldots a_n c_n}{n(n+1)} \qquad (5)$$

We have

$$[a_1 c_1 \left(\frac{1}{1.2} + \frac{1}{2.3} + \cdots \ldots\right) + a_2 c_2 \left(\frac{1}{2.3} + \frac{1}{3.4} + \cdots\right) + \cdots +$$

$$a_k c_k \left(\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots \ldots\right) + \cdots]$$

$$= \sum_{k=1}^{\infty} a_k c_k \left(\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \cdots \ldots\ldots\ldots\ldots..\right)$$

$$= \sum_{k=1}^{\infty} a_k c_k \sum_{n=k}^{\infty} \left(\frac{1}{n(n+1)}\right) \qquad (6)$$

We have $\sum_{n=k}^{\infty} \left(\frac{1}{n(n+1)}\right) = \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= \left(\frac{1}{k} - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \ldots\ldots\ldots\ldots.$$

$$= \frac{1}{k} \qquad (7)$$

Therefore from (5) , (6) and (7) , we get

$$\sum(a_1 a_2 \ldots\ldots\ldots.. a_n)^{\frac{1}{n}} \leq \sum_{k=1}^{\infty} a_k \frac{c_k}{c_k}$$

$$= \sum_{k=1}^{\infty} a_k \left(\frac{k+1}{k}\right)^k \qquad by\ (3)$$

$$= \sum_{k=1}^{\infty} a_k \left(1 + \frac{1}{k}\right)^k$$

$$\leq e \sum_{k=1}^{\infty} a_k \qquad \left(as \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e\right).$$

Therefore $\left(1 + \frac{1}{k}\right)^k \leq e \quad \forall\ k \geq 1.$

Hence $\sum(a_1 a_2 \ldots\ldots\ldots.. a_n)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} a_n$.

This shows that $\sum(a_1 a_2 \ldots\ldots\ldots.. a_n)^{\frac{1}{n}}$ is convergent as $\sum a_n$ is convergent .

Conditional and Absolutely Convergence.

Definition 1.2. A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ is convergent.

Conditional Convergence 1.3. A series which is convergent but not absolutely convergent is said to conditional convergent.

That is**,** if $\sum a_n$ is convergent but $\sum|a_n|$ is not convergent .

Then, $\sum a_n$ is conditional convergent.

Example 1.1. Consider a series $\sum\left(\frac{-1)^{n}}{n^2}\right)$.

Then, $\sum\left(\frac{-1)^{n}}{n^2}\right)$ is convergent.

Also, $\sum\left|\left(\frac{-1)^{n}}{n^2}\right)\right| = \sum\frac{1}{n^2}$ is convergent .

Therefore , $\sum\left(\frac{-1)^{n}}{n^2}\right)$ is absolutely convergent .

Example 1.2. Consider a series $\sum\left(\frac{-1)^{n}}{n}\right)$.

Then, $\sum\left(\frac{-1)^{n}}{n}\right)$ is convergent .

But, $\sum\left|\left(\frac{-1)^{n}}{n}\right)\right| = \sum\frac{1}{n}$ is not convergent.

Therefore , $\sum\left(\frac{-1)^{n}}{n}\right)$ is conditional convergent .

Theorem 1.2. Every absolutely convergent series is convergent.

or The convergence of $\sum|a_n|$ implies the convergence of $\sum a_n$.

Proof: Suppose $\sum |a_n|$ is convergent .

Hence for every $\epsilon > 0$, by Cauchy′s general principal of convergence.

There exists a positive integer m such that ,

$$\left||a_{n+1}| + |a_{n+2}| + \cdots \ldots .. + |a_{n+p}|\right| < \epsilon \quad \forall\, n \geq m \text{ and } p$$

≥ 1 (8)

Also, for all $\forall\, n \geq m$ and $p > 1$,we have

$|a_{n+1} + a_{n+2} + \cdots \ldots \ldots \ldots \ldots + a_{n+p}|$

$$\leq |a_{n+1}| + |a_{n+2}| + \cdots \ldots .. + |a_{n+p}| < \epsilon \qquad by\ (8)\ .$$

Therefore by Cauchy$'$ criteria , $\sum a_n$ is convergent .

Remark 1. Divergence of $\sum |a_n|$ does not imply divergence of $\sum a_n$.

Example 1. 3. The given series

$$\sum \left|\left(\frac{-1)^n}{n}\right)\right| \text{ is divergent .}$$

But , the series $\sum \left(\frac{-1)^n}{n}\right)$ is convergent.

This shows that the converse of above theorem is not true in general .

Test for Series of arbitrary terms.

We now consider arbitrary term series which are convergent (but not necessarily absolutely convergent) and obtain tests for their convergence. We first prove an important lemma, due to Abel's.

Lemma 1.1. If b_n is positive , monotonic decreasing function and if A_n is bounded , then the series $\sum A_n\,(b_n - b_{n-1})$ is absolutely convergent.

Proof: Since A_n is bounded .

Therefore there exists a positive number k, such that ,

$|A_n| \leq k, \quad \forall\, n$.

Thus, $\sum_{n=1}^{m} |A_n(b_n - b_{n+1})|$

$$\sum_{n=1}^{m} |A_n||(b_n - b_{n+1})| = \sum_{n=1}^{m} |A_n|(b_n - b_{n+1}) \qquad (\because\ b_{n+1} \leq b_n)$$

$$\leq \sum_{n=1}^{m} k(b_n - b_{n+1})$$

$$= k\,(b_n - b_{n+1}) < k\, b_1.$$

Thus, the sequence of partial sums of positive term series,

$\sum_{n=1}^{m} |A_n(b_n - b_{n+1})|$ is bounded above by kb_1, so that

$\sum_{n=1}^{m} |A_n(b_n - b_{n+1})|$ is convergent .

Hence $\sum A_n((b_n - b_{n+1})$ is absolutely convergent .

Theorem Abel's test 1.3.

If b_n is positive monotonic decreasing and if $\sum u_n$ is a convergent series, then the series $\sum u_n b_n$ is also convergent.

Proof: Let $V_n = u_n b_n$ and

$$S_n = \sum_{i=1}^{n} u_i \quad , \quad V_n = \sum_{i=1}^{n} v_i \ .$$

Then ,

$$V_n = u_1 b_1 + u_2 b_2 + \cdots + u_n b_n$$

$$= S_1 b_1 + (S_2 - S_1)b_2 + (S_3 - S_2)b_3 + \cdots ..(S_n - S_{n-1})b_n$$

$$= S_1(b_1 - b_2) + S_2(b_2 - b_3) \ldots \ldots . + S_{n-1}(b_{n-1} - b_n) + S_n b_n$$

$$= \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_n b_n \qquad (9)$$

Since, $\sum u_n$ *is convergent* . Therefore, the sequence $< S_n >$ *is also convergent* .

Also , $\mathrm{b_n}$ is positive and monotonic decreasing function .

Therefore by above lemma , the series $\sum S_n(b_n - b_{n+1})$ is absolutely convergent and hence the partial sums $\sum_{i=1}^{n-1} S_i(b_i - b_{i+1})$ tends to finite limit as $n \to \infty$.

Also, since b_n monotonic decreasing and bounded below by 0.

Therefore $< \mathrm{b_n} >$ is convergent and so b_n tends to a finite limit as $n \to \infty$.

Hence $S_n b_n$ tends to finite limit as $n \to \infty$.

By using the above result we find from (ix) *that* V_n tends to finite limit as $n \to \infty$.

That is the sequence $< V_n >$ of partials sums of $\sum V_n$ converges .

Consequently the series $\sum V_n$ *or* $\sum u_n b_n$ converges .

Remark. A convergent series $\sum u_n$ remain convergent if its terms are multiplied by$'a_n'$ where a_n is bounded and monotonic decreasing .

Theorem Drichlet's Test 1.4.

If b_n is positive monotonic decreasing with limit 0 and if for the series $\sum u_n$. The sequence $\{S_n\}$ of partial sums of $\sum u_n$ is bounded, then the series $\sum u_n\, b_n$ is convergent.

Proof. Let $S_n = \sum_{i=1}^{n} u_i$, $V_n = u_n b_n$

and $V_n = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} u_i\, b_i$.

Then , as before

$$V_n = \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_n b_n \qquad (10)$$

Since S_n is bounded and b_n is positive and monotonic decreasing .

Therefore by above Lemma $\sum_{i=1}^{n} S_i\ (b_i - b_{i+1})$ tends to finite limit as $n \to \infty$.

Also, since $b_n \to 0$ and $n \to \infty$, S_n is bounded.

Therefore, $S_n b_n \to 0 \quad as \;\; n \to \infty$.

Using the above result we find from (x) that V_n tends to finite limit as $n \to \infty$ and hence the series $\sum V_n \;= \sum u_n b_n$ converges.

Example 1.4. Show that the series

$$0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \cdots \quad \text{is convergent .}$$

Solution. Take $\qquad \sum V_n = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots \qquad (11)$

$and \qquad \sum u_n = 0 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{3}{4} \ldots..$

Since series (11) is convergent and the $< u_n >$ *is monotonic and bounded*.

Therefore $\sum V_n u_n = 0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \cdots \ldots$ is convergent.

Example 3.5. Test for the convergence of series, $\sum \left(\frac{(n^3 +)^{\frac{1}{3}} - n}{logn} \right)$.

Solution. Let $\quad u_n = \; \{ \left(n^3 + \right)^{\frac{1}{3}} - n \}$ and $\quad b_n = \frac{1}{logn}$.

Since $\sum u_n$ is convergent and $\{b_n\}$is positive , monotonic decreasing sequence tends to 0 as $n \to \infty$.

Therefore $\sum u_n v_n$ is convergence .

Hence given series convergent .

Rearrangement of Terms.

If the terms of finite sum are rearranged then the sum of finite series remains same. But if the term of infinite series are rearranged, then the sum of infinite series varies.

Example. Consider the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \text{ converges to the sum } S.$$

But , if terms of above series rearranged so that each positive term is followed by two negative terms, then the series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots \text{ converges to } \frac{1}{2}S.$$

Another rearrangement of above series say

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots \dots \dots \dots .\right)-\left(1+\frac{1}{3}+\frac{1}{5}+\cdots \dots \dots \dots \dots \dots \dots \dots .\right) \text{ diverges.}$$

Thus, if the series $\sum u_n$ is convergent, then the rearrangement of this series may diverge.

Rieman's Theorem 1.5. By an appropriate rearrangement of the terms a conditional convergent series $\sum u_n$ can be

(i) Converge to any number σ or

(ii) Diverge to $+\infty$

(iii) Diverge to $-\infty$

(iv) Can oscillate finitely

(v) Can oscillate infinitely.

Proof. Let $$a_n = \begin{cases} u_n, & if\ u_n \geq 0 \\ 0, & if\ u_n < 0 \end{cases}$$

and $$b_n = \begin{cases} -u_n, & if\ u_n < 0 \\ 0, & if\ u_n \geq 0 \end{cases} .$$

Then, clearly a_n and b_n are non − negative and

$$u_n = a_n - b_n$$

$$|u_n| = a_n + b_n \tag{12}$$

Since $\sum u_n$ *is* conditional convergent .

Therefore $\sum|u_n|$ diverges and hence from (12) , at least one of the series $\sum a_n$, or $\sum b_n$ diverges.

Again, Since $\sum u_n$ is convergent .

Therefore from (12), it follows that the two series $\sum a_n$,

$\sum b_n$ either both converges or both diverges .

Thus, $\sum a_n$ and $\sum b_n$ both diverges .

Also, $a_n \to 0, b_n \to 0$ $\quad$ $(because\ \ u_n \to 0\ \ as\ n \to \infty)$.

(i) We shall first show that a rearrangement $\sum V_n\ of\ \sum u_n$ can be found which converges to any number σ.

Let n_1 be the least number of terms of $\sum a_n$, such that

$$a_1 + a_2 + \cdots + a_{n1} \quad > \quad \sigma$$

Let m_1 be the least number of terms of the series $\sum b_n$, such that

$$a_1 + a_2 + \cdots + \cdots + a_{n_1} - b_1 - b_2 - \cdots - b_{m_1} < \sigma$$

Again, let n_2 be the number of terms of $\sum a_n$, other than previous, such that

$$a_1 + a_2 + \cdots + a_{n_1} - b_1 - b_2 \dots . -b_{m_1} +$$

$$+a_{n_{1+1}} + a_{n_{1+2}} + \cdots + a_{n_{n1+n2}} > \sigma$$

Let m_2 be the least no. of next terms of $\sum b_n$, such that

$$a_1 + a_2 + \cdots + a_{n_1} - b_1 - b_2 - \cdots - b_{m_1} +$$

$$+a_{n_{1+1}} + a_{n_{1+2}} + \cdots + a_{n_{n1+n2}} - b_{m_{1+1}} - b_{m_{1+2}} - \cdots - b_{m_{1+m2}} < \sigma$$

The process may be continued indefinitely. The process indicated above is always possible, because of the divergence of two series $\sum a_n$, and $\sum b_n$.

Let $\sum V_n$ be the rearranged series and $\{\sigma_n\}$its sequence of partial sums.

Clearly, $\sigma_{n_1} > \sigma$, $\sigma_{n_{1+m1}} < \sigma_{n_{1+m1+m2}} > 6$, $6_{n_{1+m1+n2+m2}} < \sigma$

Therefore it can be easily shown that the sequence $\{\sigma_n\}$ converges to σ.

This implies that

(i) The rearrangement series $\sum V_n$ converges to σ.

(ii) We shall now show that a suitable rearrangement of $\sum a_n$, can be found which diverges to $+\infty$.

Let us consider the rearrangement

$$a_1 + a_2 + \ldots\ldots a_{m_1} - b_1 + b_{m_{1+1}} + \cdots$$

$$+ a_{m_2} - b_2 + a_{m_{2+1}} + \cdots$$

in which a group of positive terms followed by single negative term.

This is certainly a rearrangement of $\sum u_n$ and let us denote it by $\sum V_n$ and its partial sum by S_n .

Now since the series
$\sum \mathrm{a_n}$ is divergent , its partial sums are therefore unbounded .

Let us choose m_1 , so large that $a_1 + a_2 + \cdots \ldots . . + a_{m_1} > 1 + b_1$

Then,

$m_2 > m_1$, so large such that

$$a_1 + a_2 + \cdots \ldots . . a_{m_1} + \cdots . + a_{m_2} > 2 + b_1 + b_2$$

And in general $m_n > m_{n-1}$ so large that

$a_1 + a_2 + \cdots + a_{mn} > n + b_1 + b_2 + \cdots + b_n$ for n=1,2,3…. .

Now since each of the partial sum $S_{m_{1+1}}, S_{m_{2+1}}, \ldots$ of $\sum V_n$ whose last term is negative term $-b_n$ is greater than $n(n = 1,2,3 \ldots \ldots ..)$.

Thus, the series $\sum V_n$ diverges to $+\infty$.

(ii) By considering the rearrangement

$$-b_1 - b_2 - \cdots .. b_{m_1} + a_1 - b_{m_{1+2}} - \cdots .. - b_{m_2} + \cdots .. \text{it can be}$$

This show that the rearrangement diverges to $-\infty$.

Other cases may similarly be proved by considering the suitable rearrangement of given series.

Example 1.6. Criticize the following paradox

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \ldots\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots ..\right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \ldots ..\right) - \left(1 - \frac{1}{2} - \cdots \ldots \ldots ..\right)$$

$$= 0$$

Hence the series converges to zero.

Multiplication of Series.

Definition 1.3. Given two series $\sum a_n$ and $\sum b_n$.

We put $C_n = \sum_{k=0}^{\infty} a_k b_{n-k} (n = 0,1,2, \ldots \ldots .)$

and call $\sum C_n$, the cauchy's product of $\sum a_n$ and $\sum b_n$.

Here $\sum a_n \sum b_n = \sum C_n$.

Let us denoted by

$$A_n = a_0 + a_1 + a_2 + \cdots \ldots + a_n$$

$$B_n = b_0 + b_1 + b_2 + \cdots \ldots . + b_n$$

and $C_n = (a_0 b_0) + (a_1 b_0 + a_0 b_1) + a_2 b_0 + a_1 b_1 + a_0 b_2) +$
$\dots (a_0 b_n + \cdots + a_n b_0)$

Note that $A_n B_n \neq C_n$.

Marten's Theorem 1.6. Suppose $\sum a_n = A$

and $\sum b_n = B$ and $\sum a_n$ is absolutely Convergent,

Then, $Lim\, C_n = AB$.

Proof. Put $A_n = a_0 + a_1 + a_2 + \cdots \dots + a_n$

$B_n = b_0 + b_1 + b_2 + \cdots .. + b_n$

and $\beta_n = B_n - B$ *where* $\beta_n \to 0$ $as\ n \to \infty$.

Then, $C_n = \sum_{k=0}^{n} a_k b_{n-k}$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots . + a_n b_0)$

$= a_0 B_n + a_1 (B_{n-1}) + \cdots .. + a_n B_o$

$= a_0(\beta_n + B) + a_1(\beta_{n-1} + B) + \cdots \dots \dots + a_n(\beta_0 + B)$

$= B\ (a_0 + a_1 + a_2 + \cdots \dots + a_n)$

$+ a_0 \beta_n + a_1 \beta_{n-1} + \cdots .. + a_n \beta_0$.

This implies that $C_n = A_n B + \gamma_n$

where $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$.

$To\ prove\ that\ C_n \to AB$.

Since $A_n B \to AB$, it is suffice to show that

$\gamma_n \to 0\ as\ \ n \to \infty$.

Put $\alpha = \sum |a_n|$.

Since $\beta_n \to 0\ \ as\ \ n \to \infty$.

Hence given $\epsilon > 0$, $we\ can$ find N $such\ that$

$|\beta_n| < \epsilon, \qquad \forall \ \ n \geq N./$

Therefore,

$$|\gamma_n| = |a_n\beta_0 + a_{n-1}\beta_1 + \cdots a_{n-N+1}\beta_{N-1} + a_{n-N}\beta_N + \cdots. +a_0\beta_n|$$

$$\leq |a_n\beta_0 + a_{n-1}\beta_1 + \cdots a_{n-N+1}\beta_{N-1}| + |a_{n-N}\beta_N + \cdots \ldots. +a_0\beta_n|$$

$$\leq |a_n\beta_0 + a_{n-1}\beta_1 + \cdots \ldots \ldots a_{n-N+1}\beta_{N-1}| + \epsilon\alpha \ .$$

Keeping N fixed and Letting $n \to \infty$ and noting that $a_n \to 0$ $as\ n \to \infty$.

We get,

$$\lim_{n\to\infty} |\gamma_n| \leq \epsilon\,\alpha \ .$$

Now letting $\epsilon \to 0$, we get

$$\lim_{n\to\infty} |\gamma_n| = 0.$$

This implies $\lim_{n\to\infty} \gamma_n = 0.$

Thus $\lim_{n\to\infty} C_n = AB.$

Uniform Continuity.

Definition 1.4. A function f on interval I is uniformly continuous if for each $\epsilon > 0$, $\exists\ \delta > 0$,such that

$$|f(x_1) - f(x_2)| < \epsilon \qquad whenever\ |x_1 - x_2| < \delta \qquad x_1\ ,\ x_2\ \in I.$$

Theorem 3.7. If f is uniformly contineous on I, then it is contineous on that interval .

Proof. Suppose f is uniformly contineous on I. Then , $\forall\ \ \varepsilon > 0, \exists\ \ \delta > 0$, such that

$$|f(x_1) - f(x_2)| < \epsilon \quad whenever\ |x_1 - x_2| < \delta \qquad x_1, x_2 \in I \qquad (13)$$

Let $a \in I$.

Then, for all $\epsilon > 0$, $\exists\ \delta > 0$, *such that*

$$|f(x) - f(a)| < \epsilon \qquad whenever\ |x - a| < \delta$$

This implies f is contineous at $x = a$. Since $'a'$ is choose arbitrary.

Therefore f is contineous on I.

Heine's Theorem 1.8. A function which is continuous on closed interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

Proof. f is contineous on [a, b].

Let if possible f be not uniformly continuous on $I = [a, b]$ then there exit $\varepsilon > 0$, such that for any $\delta > 0$, there are numbers $x, y \in I$, for which

$$|f(x) - f(y)| \nless \varepsilon \qquad \text{whenever} \qquad |x - y| < \delta\ .$$

Hence for each positive n, we can find $x_n,\ y_n\ \varepsilon\ I$, such that

$$|f(x_n) - f(y_n)| \geq \epsilon \qquad (14)$$

whenever $|x_n - y_n| < \delta$.

Since $\{x_n\}$, $\{y_n\}$ being sequence in I , they are bounded and therefore each has atleast one limit point say a_1 and a_2 respectively .

Since I is closed set.

Therefore $a_1, a_2 \in I$.

Since a_1 is limit point of $\{x_n\}$, there exits a convergent subsequence $\{x_{nk}\}$ of $\{x_n\}$, such that

$$x_{nk} \to a_1\ as\ \ k \to \infty.$$

Similarly there exists a convergent subsequence $\{y_{nk}\}\, of\, \{y_n\}$, such that

$$y_{nk} \to a_2 \, as \;\; k \to \infty.$$

Again from (14)

$$|f(x_{nk}) - f(y_{nk})| \nless \epsilon$$

Whenever $\quad |(x_{nk}) - (y_{nk})| < \frac{1}{n_k} \leq \frac{1}{k}$ (15)

Second inequality shows that

$$\lim_{k\to 0} x_{nk} = \lim_{k\to\infty} y_{nk} \; .$$

This implies

$$a_1 = a_2 = a \; .$$

But, $\{\, f(x_{nk})\}\;\, and \;\, \{f(y_{nk})\}$ Converges to two different numbers by (14) which contradicts to the fact that f is continuous on $I = [a, b]$.

Thus our supposition is wrong.

Hence f is uniformly continuous on $I = [a, b]$.

Exercise. Show that (i) $f(x) = x^2$ is uniformly contineous on $]0,1]$.

(ii) $f(x^2)$ is uniformly contineous on $[-1,1]$.

(iii) $f(x) = sinx$ is uniformly contineous on $[0, \infty)$.

Darboux's Theorem 1.9. If f is derivable on $[a, b]$ and $f'(a) \neq f'(b)$, then for any number k between $f'(a)$and $f'(b)$, $\exists\, c \in (a, b)$, such that

$$f'(c) = k \; .$$

Proof. Suppose $f'(a) < k < f'(b)$.

Consider a function $g(x) = f(x) - k(x)$.

Then, $g'(a) = f'(a) - k < 0$

and $g'(b) = f'(b) - k > 0$.

By definition, $\underset{x \to a}{Lim} \frac{g(x)-g(a)}{x-a} = g'(a)$

and $\underset{x \to b}{Lim} \frac{g(x)-g(b)}{x-b} = g'(b)$.

Thus, $\forall \epsilon > 0$, $\exists \quad \delta_1 \, \delta_2$, such that

$$\left|\frac{g(x)-g(a)}{x-a} - g'(a)\right| < \epsilon \qquad \text{when} \qquad |x-a| < \delta_1$$

and

$$\left|\frac{g(x)-g(b)}{x-b} - g'(b)\right| < \epsilon \qquad \text{when} \qquad |x-b| < \delta_2 .$$

Let $\delta = min\{\delta_1 \, \delta_2\}$.

Then $\left|\frac{g(x)-g(a)}{x-a} - g'(a)\right| < \epsilon$ *and* $\left|\frac{g(x)-g(b)}{x-b} - g'(b)\right| < \epsilon$

whenever $|x-a| < \delta$, $|x-b| < \delta$.

This implies

$$g'(a) - \epsilon < \frac{g(x)-g(a)}{x-a} < g'(a) + \epsilon,$$

This implies that

$$g'(b) - \epsilon < \frac{g(x)-g(b)}{x-b} < g'(b) + \epsilon$$

$$\text{whenever} \quad |x-a| < \delta, \; |x-b| < \delta \qquad (16)$$

Choose ϵ so small , such that

$$\left.\begin{array}{l} g'(a) + \epsilon < 0 \\ \text{and} \\ g'(b) - \epsilon > 0 \end{array}\right\} \qquad (17)$$

Thus from (1) and (2), we get

$$g(a+h) < g(a), \quad g(a-h) > g(a)$$

and

$$g(b+h) > g(b), \quad g(b-h) < g(b) \qquad \text{where } 0 < h < \delta \qquad (18)$$

Since g is derivable on $[a, b]$.

Therefore continuous on $[a, b]$and attains supremum in $[a, b]$.

Clearly by (18) g has no supremum at a or at b.

Then there exist $c \in (a, b)$, such that

$$g(c) = sup\{g(x)\}$$

Claim: $g'(c) = 0$.

If possible $g'(c) > 0$.

Then, $g(c+h) > g(c)$ as before which is not possible.

Also if, $g'(c) < 0$.

Then , $g(c-h) > g(c)$ as before which is again not possible .

Thus, $g'(c) = 0$.

This implies that

$$f'(c) = k \qquad \text{where} \qquad c \in (a, b).$$

Hence the theorem follows.

Corollary. If f is derivable in $[a, b]$ such that $f'(a)$ and $f'(b)$are opposite in sign , then there exits $0 \in (a, b)$, such that $f'(0) = 0$.

Theorem 1.10. If $\overline{\lim}|a_n|^{\frac{1}{n}} = \frac{1}{R}$, , then

The sreies $\sum a_n x^n$ is convergent for $|x| < R$

and diverges for $|x| > R$.

Proof. We have

$$\overline{\lim}_{n\to\infty}|a_n x^n|^{1/n} = \overline{\lim}|a_n|^{\frac{1}{n}}.|x| = \frac{|X|}{R} .$$

Therefore $\sum a_n x^n$ converges, if

$\frac{|X|}{R} < 1$ and diverges if $\frac{|x|}{R} > 1$.

Thus, $\sum a_n x^n$ converges, if $|x| < R$

and diverges if $|x| > R$ *when* $R = \frac{1}{\overline{\lim}|a_n|^{\frac{1}{n}}}$.

Definition 1.5. In view of above theorem, the radius of convergence of power series $\sum a_n x^n$ is defined as

$$\frac{1}{\overline{\lim}|a_n|^{\frac{1}{n}}}, \quad when \quad \overline{\lim}|a_n|^{\frac{1}{n}} > 0$$

$$= \infty \quad , when \quad \overline{\lim}|a_n|^{\frac{1}{n}} = 0$$

$$= 0 \quad , when \quad \overline{\lim}|a_n|^{\frac{1}{n}} = \infty$$

Abel's Limit Theorem 1.11. (First form, at Centre).

If the power series converges at the end point $x = R$ of the interval of the convergence $-R < x < R$, *then it is uniformly convergent in* $[0, R]$.

Proof. We shall show that under the given assumptions cauchy's criteria for uniform convergence is satisfied in $[0, R]$. This will imply the uniform convergence of power series $\sum a_n x^n$ on $[0, R]$.

Let $\quad S_{n,p} = \ a_{n+1}R^{n+1} + \cdots . + a_{n+p} \qquad for \ \ p = 1,2,3, \dots \dots$

Then, obiviously

$$a_{n+1}R^{n+1} = S_{n,1}$$

$$a_{n+2}R^{n+2} = S_{n,2} - S_{n,1}$$

$$\vdots$$

$$a_{n+p}\, R^{n+p} = S_{n.p} - S_{n,p-1} \tag{19}$$

Let $\epsilon > 0$ *be given* . Since the series

$\sum a_n R^n$ is convergent , therefore by Cauchy$'$sgeneral principal of convergence

there exists an integer N such that for $n \geq N$

$$|S_{n.q}| < \epsilon, \qquad \forall \ q = 1,2,3 \dots ., \tag{20}$$

Taking into account that

$$\left(\frac{X}{R}\right)^{n+p} \leq (\frac{x}{R})^{n+p-1} \leq \cdots \leq \ (\frac{x}{R})^{n+1} \qquad for \quad 0 \leq x \leq R$$

Using (19) and (20), we have

$$For\ n \geq N$$

$$|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots \dots \dots \dots \dots . + a_{n+p}x^{n+p}|$$

$$= \left|a_{n+1}R^{n+1}(\frac{x}{R})^{n+1} + a_{n+2}R^{n+2}(\frac{x}{R})^{n+2} + \cdots \dots \dots + a_{n+p}R^{n+p}(\frac{x}{R})^{n+p}\right|$$

$$= \left|S_{n,1}(\frac{x}{R})^{n+1} + (S_{n,2} - S_{n,1})(\frac{x}{R})^{n+2} + \cdots \dots \dots + (S_{n,p}S_{n,p-1})(\frac{x}{R})^{n+p}\right|$$

$$= \left| S_{n,1}[(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}] + S_{n,2}[(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}] + \cdots \ldots \ldots + S_{n,p-1}[(\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p}] + S_{n,p}(\frac{x}{R})^{n+p} \right|$$

$$\leq |S_{n,1}| \ \{((\frac{x}{R})^{n+1} - \left(\frac{x}{R}\right)^{n+2}\} + |S_{n,2}|[(\frac{x}{R})^{n+2} - \left(\frac{x}{R}\right)^{n+3}\Big] + \ldots\ldots$$

$$\ldots\ldots + |S_{n,p-1}|[(\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p}] + |S_{n,p}|(\frac{x}{R})^{n+p}$$

$$< \ \epsilon \ \ \{ (\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2} + (\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3} + \cdots$$

$$\ldots + (\frac{x}{R})^{n+p-1} - \left(\frac{x}{R}\right)^{n+p}\Big\} \qquad \textit{by (20)}$$

$$= \ \epsilon \, (\frac{x}{R})^{n+1} \ \leq \ \epsilon \qquad \forall \ n \geq N, \ p \geq 1, \forall \ x \in [0, R] \, .$$

Thus $\forall \ \epsilon > 0, \ \exists \ N \textit{ such that}$

$$\left| a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots \ldots \ldots \ldots \ldots + a_{n+p}x^{n+p} \right| < \epsilon$$

$$\forall \ n \geq N, \ p \geq 1, \ x \in [0, R] \, .$$

Hence by Cauchy’s criteria, the series converges uniformly on [0, R].

Abel’s Theorem 1.12. (Second Form).

If $\sum a_n x^n$ be power series with finite radius of convergence R and let

$f(x) = \sum a_n x^R$, $-R < x < R$. If the $\sum a_n R^n$ converges, then

$$\lim_{x \to R-0} f(x) = \sum a_n R^n.$$

Proof. Let us first show that there is no loss of generality in taking R=1

Put $x = Ry$, so that

$$\sum a_n x^n = \sum a_n R^n y^n$$

$$= \sum b_n y^n \qquad \text{where } b_n = a_n R^n$$

It is power series with radius of convergence R'

where $$R' = \frac{1}{\overline{\lim}|a_n R^n|^{\frac{1}{n}}} = \frac{1}{R}.\frac{1}{\overline{\lim}|a_n|^{\frac{1}{n}}}$$

$$= \frac{1}{R}.R = 1 .$$

Thus it suffice to prove the following:

Let $\sum a_n x^n$ be power series with unit radius of convergence and let

$$f(x) = \sum a_n x^n , \quad -1 < x < 1 .$$

If the $\sum a_n$ converges, then

$$\lim_{x \to 1_-} f(x) = \sum a_n$$

Let $S_n = a_0 + a_1 + \cdots \ldots . + a_n$

$$S_{-1} = a_0$$

and let $\sum_{n=0}^{\infty} a_n = S = \sum a_n,$

Then, $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{m}(s_n - s_{n-1})x^n$

$\sum_{n=0}^{m} s_n x^n - \sum_{n=0}^{m} s_{n-1} x^n$

$= \sum_{n=0}^{m-1} s_n x^n + s_m x^m - \sum_{n=0}^{m} s_{n-1} x^n$

$= \sum_{n=0}^{m-1} s_n x^n - x \sum_{n=0}^{m} s_{n-1} x^{n-1} + s_m x^m$

$= \sum_{n=0}^{m-1} s_n x^n - \sum_{n=0}^{m-1} s_n x^n + s_m x^m$

$=(1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m$ *for* $|x| < 1, when \ m \to \infty,$

Since $s_m \to s$ and $x^m \to 0$, we get

$$f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n, for \ 0 < x < 1 \qquad (21)$$

Again since $s_n \to s$, for $\epsilon > 0$. *there exists* N, *such that*

$$|S_n - s| < \frac{\epsilon}{2} \qquad \forall \ n \geq N \qquad (22)$$

Also, $(1-x) \sum_{n=0}^{\infty} x^n = 1$, *for* $|x|<1$ *(23)*

$$\left(((1-x)\sum x^n) = \sum_0^{\infty} x^n - \sum_0^{\infty} x^n - \sum_0^{\infty} x^{n+1} = 1)\right)$$

Hence for $n \geq N$, we have f*or* $0 < x < 1$

$$|f(x) - s| = |(1-x) \sum_{n=0}^{\infty} s_n x^n - s| \qquad using \ (21)$$

$$= |(1-x) \sum_{n=0}^{\infty} s_n x^n - (1-x) \sum_{n=0}^{\infty} x^n s| \qquad by\ (23)$$

$$= |(1-x) \sum_{n=0}^{\infty} (s_n - s) x^n|$$

$$\leq \quad (1-x)\sum_{n=0}^{\infty}|(s_n - s)|x^n$$

$$= \quad (1\text{-}x)\{\sum_{n=0}^{N}|(s_n - s)|x^n + \sum_{n=N+1}^{\infty}|(s_n - s)|x^n\}$$

$$< (1\text{-}x)\left\{\sum_{n=0}^{N}|(s_n - s)|x^n + \frac{\epsilon}{2}\sum_{n=N+1}^{\infty} x^n\right\} \quad \textit{by (22)}$$

$$\leq \quad (1\text{-}x)\sum_{n=0}^{N}|s_n - s|x^n + \frac{\epsilon}{2}$$

But for fixed *N*, $(1\text{-}x)\sum_{n=0}^{N}|S_n - S|\, x^n$ is a positive continuous function of X having zero value at x=1.

Therefore, *∃ δ >0*, such that *for* 1-δ < x < 1,

$$(1\text{-}x)\sum_{n=0}^{N}|S_n - S|\, x^n \quad < \frac{\epsilon}{2}$$

$\therefore \qquad |f(x) - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad$ for $1 - \delta < x < 1$.

Hence $\qquad \lim_{x\to 1-} f(x) = S = \sum_{n=0}^{\infty} a_n$.

This completes the theorem.

In this section we shall study the theory of Riemann-Stieltjes integration which is the generalization of Riemann theory of Integration. It may be stated once for all that, unless otherwise stated, all functions will be real valued and bounded on the domain of definition. The function ɑ will always be monotonic increasing.

Definition & Existence of Riemann–Steiljes Integral (RS-Integral).

Definition 2.1: Let *f* and α be bounded functions on $[a,b]$ and α be monotonic increasing function on $[a,b]$, $b \geq a$

Corresponding to any partition

$$a = x_0 \leq x_1 \leq x_2 \leq . . . \leq x_{n-1} \leq x_n = b .$$

We write,

$$\Delta x_i = x_i - x_{i-1} \quad for \quad i = 1, 2, . . . , n .$$

It is clear that, $\Delta x_i \geq 0$.

As ɑ be a monotonically is increasing function on $[a,b]$. Since $\alpha(a)$ *and* $\alpha(b)$ are finite, it follows that α is bounded on $[a,b]$.

Corresponding to each partition P of $[a,b]$, we have

$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \quad for \quad i = 1, 2, . . . , n .$$

It is clear that $\Delta \alpha_i \geq 0$. For any real function f which is bounded on $[a,b]$, we have

$$U(P, f, \alpha) = \sum_{i=1}^{n} M \Delta \alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m \Delta \alpha_i$$

where M_i , m_i are the bounds (supremum and infimum of f respectively over $\Delta x_i = [x_{i-1}, x_i]$.

If *m* and *M* are the lower and the upper bounds *f* on $[a, b]$, we have

Then, $m \le m_i \le M_i \le M$

This implies that

$$m \sum_{i=1}^{n} \Delta\alpha_i \le \sum_{i=1}^{n} m_i \, \Delta\alpha_i \le \sum_{i=1}^{n} M_i \, \Delta\alpha_i \le M \sum_{i=1}^{n} \Delta\alpha_i .$$

This implies

$$m\sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})] \le L(p, f. \alpha) \le U(p, f, \alpha) \le M \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]$$

This gives

$$m\,[\alpha(b) - \alpha(a)] \le L(p, f, \alpha) \le U(p, f, \alpha) \le M\,[\alpha(b) - \alpha(a)] \qquad (1)$$

Since we have infinite number of partitions on $[a, b]$ and for every partition, we have upper sum and lower sum.

Let S_1 is the set of upper sums. And S_2 be the set of lower sums

That is

$$S_1 = \{U(p, f, \alpha) : p \in p\,[a, b]\},$$

and

$$S_2 = \{L(p, f, \alpha) : p \in p\,[a, b]\}.$$

From (1), S_1 and S_2 are bounded sets.

Therefore S_1 has greatest lower bounded and S_2 has least upper bound.

Let *g. l. b* $(S_1) = \int_a^{-b} f(x)d\alpha(x)$,

and *l. u. b* $(S_2) = \int_{\_a}^{b} f(x)d\alpha(x)$

That is $\int_a^{-b} f d\alpha = Inf.\{u(p, f, \alpha): p \in p[a, b]\},$

and $\int_{\_a}^{b} f d\alpha \quad = \quad Sup.\{l(p,f,\alpha) : p \epsilon p[a,b]\}$

These are respectively called upper and lower the integrals of f with respect to α .

These two integrals may or may not be equal. In case these two integrals are equal.

$$\therefore \quad \int_{a}^{-b} f\, d\alpha \quad = \quad \int_{\_a}^{b} f\, d\alpha\ .$$

We say f is interable with respect to α in the Riemann sence and , we write

$f \in R_\alpha[a,b]$ or simply $f \epsilon R_\alpha$ and the common value is denoted by

$$\int_a^b f\, d\alpha$$

or sometimes by $\int_a^b f(x)\, d\alpha(x)$

and is called the Riemann – Stieltjes integral of f with respect to over $[a, b]$.

If $\int_a^b f\, d\alpha$ exists, then we say that f is integrable with respect to α ,

in the sense of Riemann, and, we write $f \in R(\alpha)$.

By taking $\alpha(x) = x$, the Riemann integral will be a special case of the Riemann –Stieljies integral.

Refinement of Partitions.

Definition 2.2. For any partition P, the length of the largest sub – interval is called norm or mesh of the partition and is denoted by $\mu(p)$, (or simply μ) and $(P) = \ max\, \Delta x_i \ , \ \ 1 \leq i \leq n$.

A partition p^* is said to refinement of p if $p \subseteq p^*$.

We also say that p^* refines P or that p^* is finer than P.

If p_1 and p_2 are two partitions, then we say that $p*$ is their common refinement if

$$p* = p_1 U\, p_2 \ .$$

Theorem 2.1. If $p*$ is a refinement of a partition p, then for a bounded function f,

(i) $L(p, f, \alpha) \leq (p*,, f, \alpha)$

(ii) $U(p*, f, \alpha) \leq U(p, f, \alpha)$

Proof. To prove (i), suppose first that p* contains just one point more than p.

Let this extra point be ξ, and suppose this point is in $\Delta x_i = [x_{i-1,}\, x_i]$,

That is, $x_{i-1} < \xi < x_i$.

As the function is bounded on entire interval [*a, b*]. It is bounded in every sub interval Δx_i ($i = 1,2,3 \ldots .n$).

Let w_1, w_2 , mi be the infimum of f in the intervals $[x_{i-1,}\, \xi]$, $[\xi, x_i]$ and $[x_{i-1}, x_i]$, respectively .

Clearly $m_i \leq W_1$, $m_i \leq W_2$.

We have

$$L\ (p*, f, \alpha) - L\ (p, f, \alpha)$$

$$=\{w_1[\alpha(\xi) - \alpha(x_{i-1}) + W_2[\alpha(x_i) - \alpha(\xi)] - m_i[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= (\ w_1 - m_i\ )[\alpha(\xi) - \alpha(x_{i-1})] + (\ w_2 - m_i\ )[\alpha(x_i) - \alpha(\xi)]$$

$$\geq 0\ .$$

If p* contains k points more than p, we repeat this reasoning k times and conclude

$$L(p^*, f, \alpha) - L(p, f, \alpha) \geq 0$$

This implies that

$$L(p*, f, \alpha) \geq L(p, f, \alpha)$$

or $$L(p, f, \alpha) \leq (p*,, f, \alpha) .$$

(ii) Home Assignment . The result follows from Theorm 2.1 (1).

Theorem 2.2. If f is bounded function on $[a, b]$ and $p_1, p_2 \in P[a, b]$, then

$$L(p_1, f, \alpha) \leq U(p_2, f, \alpha).$$

Proof. Let $P = p_1 \cup p_2$ be the common refinement of p_1 and p_2 .
Then from the above theorem , we have

$$L(p_1, f, \alpha) \leq L(p, f, \alpha) \leq U(p, f, \alpha) \leq U(p_2, f, \alpha).$$

This gives

$$L(p_1, f, \alpha) \leq U(p_2, f, \alpha) .$$

Hence theorem 2.2 follows.

Theorem 2.3. If f *be* a bounded function on [*a, b*], then

$$\int_{\_a}^{b} f\, d\alpha \leq \int_{a}^{-b} f\, d\alpha .$$

Proof. We know that

$$U(p_1, f, \alpha) \geq L(p, f, \alpha) \qquad for\ all\ p_1, p_2 \epsilon P[a, b]. \qquad (2)$$

We first keep 'p_2' fixed and vary p_1, then (1) gives

$$g.l.b_{p_1 \epsilon P[a,b]} (U(p_1, f, \alpha)) \geq L(p_2, f, \alpha)$$

implies that $$\int_{a}^{-b} f\, d\alpha \geq L(p_2, f, \alpha) \qquad (3)$$

Now we vary p_2 , then (2) gives

$$\int_{a}^{-b} f\, d\alpha \geq l.u.b_{p_2 \epsilon P[a,b\}}(L(p_2, f, \alpha))$$

This gives $\overline{\int_a^b} f\, d\alpha \geq \underline{\int_a^b} f\, d\alpha$.

This gives $\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$.

Hence the Theorem follows.

Theorem 2.4. If $f \in R(\alpha)$, then

$$m\,[\alpha(b) - \alpha(a)] \leq \int_a^b f\, d\alpha \leq M\,[\alpha(b) - \alpha(a)] .$$

Proof . Let p be any partition of [a, b], then

$$U(p, f, \alpha) \geq \overline{\int_a^b} f\, d\alpha , \tag{4}$$

and $$L(p, f, \alpha) \leq \underline{\int_a^b} f\, d\alpha \tag{5}$$

Also $$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha , \tag{6}$$

and

$$m[\alpha(b) - \alpha(a)] \leq L(p, f, \alpha) \leq U(p, f, \alpha) \leq M[\alpha(b) - \alpha(a)] \tag{7}$$

Thus, (4), (5), (6), and (7) gives

$$[m\,[\alpha(b) - \alpha(a)] \leq L(p, f, \alpha) \leq \int_a^b f d\alpha \leq U(p,f,\alpha) \leq M\,[\alpha(b) - \alpha(a)].$$

This implies that

$$[m\,[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M\,[\alpha(b) - \alpha(a)].$$

Example 2.1: - If p* is refinement of p, then

$$U(p*, f, \alpha) - L(p*, f, \alpha) \leq U\,(p, f, \alpha) - L(p, f, \alpha)$$

Solution: Home Assignment.

A Condition of Integrability.

Theorem 2.5. A function $f \in R_\alpha[a, b]$, iff for every $\epsilon > 0$, *there exists a* partition p of $[a, b]$, such that

$$U(p\ ,f,\alpha)-L(p,f,\alpha)<\ \epsilon.$$

Proof. (Necessary part). Let f ∈ $R_\alpha[a,b]$

Therefore $\int_a^{-b} f d\alpha = \int_{\_a}^{b} f d\alpha = \int_a^b f\, d\alpha$ (8)

Let ϵ be any number .Since $\int_a^{-b} f d\alpha$ and $\int_{\_a}^{b} f d\alpha$ are the infimum and supremum of set of upper sums and set of lower sums respectively.

Therefore there exits partitions $p_1,\ p_2 \in\ P[a,\ b]$, such that

$$\int_a^{-b} f d\alpha + \frac{\varepsilon}{2} > U\ (p_1,\ f,\ \alpha) \quad (9)$$

and

$$\int_{\_a}^{b} f d\alpha - \frac{\varepsilon}{2} < L\ (p_2,\ f,\ \alpha) \quad (10)$$

Let p= $p_1 \cup p_2$, be common refinement of p_1 and p_2, then

$$L(p_1,f,\alpha) \geq L(p_2,f,\alpha) \quad (11)$$

$$U(p,\ f,\ \alpha) \leq\ U(p_1,f,\alpha) \quad (12)$$

Now, (9), (10), (11), and (12) gives

$$U\ (p_1\ ,\ f,\ \alpha) <\ \int_a^{-b} f d\alpha + \frac{\varepsilon}{2}$$

and

$$L\ (p_1,\ f,\ \alpha) <\ \int_{\_a}^{b} f\ d\alpha\ - \frac{\varepsilon}{2}$$

Since p is the common refinement of the partitions $of\ p_1, p_2$, we have

$$U\ (p\ ,\ f,\ \alpha) <\ \int_a^{-b} f d\alpha + \frac{\varepsilon}{2}$$

and $L\ (p_1,\ f,\ \alpha) <\ \int_{\_a}^{b} f\ d\alpha\ - \frac{\varepsilon}{2}$ by (8).

This gives $U(p, f, \alpha) - L(p, f, \alpha) < \epsilon$.

This follows necessary part.

(Sufficient part). Let $\epsilon > 0$, and p be a partition , for which

$$U(p, f, \alpha) - L(p, f, \alpha) < \epsilon \ .$$

For any partition p we know that

$$L(p, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(p, f, \alpha) \ .$$

This implies $\overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha \leq U(p, f, \alpha) - L(p, f, \alpha) < \epsilon$.

This implies that

$$\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \epsilon \qquad (13)$$

Since $\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \geq 0$ and $\epsilon > 0$, be any number.

Therefore from (13) , we must have

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha \ .$$

This gives $f \in R_\alpha[a, b]$.

Thus the sufficient part follows and completes the proof of the result.

Theorem 2.6. If $f_1 \epsilon R_\alpha$ and $f_2 \epsilon R_\alpha$ over $[a, b]$, then

$$f_1 + f_2 \epsilon R_\alpha[a, b] \text{ and } \int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \ .$$

Proof. Let $f = f_1 + f_2$.

Since f_1 and f_2 are bounded.

Therefore $f_1 + f_2 = f$ is also bounded on $[a. b]$

If $P = \{a = x_0, x_1, x_2, \ldots. x_n = b\}$ be any partition of $[a, b]$ and let

m_i' ,M_i' , m_i'' , M_i'' and m_i, M_i are the infimums and supremums of f_1, f_2 and *f* respectively on Δx_i , then

$$m_i' + m_i'' \leq m_i \leq M_i \leq Mi' + M_i''.$$

Multiplying by $\Delta\alpha_i$, we get

$$m_i' \, \Delta\alpha_i \leq m_i'' \, \Delta\alpha_i \leq m_i \, \Delta\alpha_i \leq M_i \, \Delta\alpha_i \leq M_i' \, \Delta\alpha_i + M_i'' \, \Delta\alpha_i \; .$$

This gives

$$\sum_{i=1}^{n} m_i' \, \Delta\alpha_i + \sum_{i=1}^{n} m_i'' \, \Delta\alpha_i \leq \sum_{i=1}^{n} m_i \, \Delta\alpha_i \leq$$

$$\sum_{i=1}^{n} M_i \, \Delta\alpha_i \leq \sum_{i=1}^{n} M_i' \, \Delta\alpha_i + \sum_{i=1}^{n} M_i'' \, \Delta\alpha_i \; .$$

This implies

$$L(p, f_1, \alpha) + L(p, f_2, \alpha) \leq L(p, f, \alpha) \leq$$

$$\leq U(p, f, \alpha)) \leq U(p, f_1, \alpha) +) \leq U(p, f_2, \alpha \qquad (14)$$

This implies that

$$U(p, f, \alpha) - L(p,f,\alpha) \leq \{ U(p, f_1, \alpha) + U(p, f_2, \alpha)\} - \{L(p, f_1, \alpha) + L(p, f_2, \alpha)\}.$$

This implies

$$U(p, f, \alpha) - L(p,f,\alpha) \leq U(p, f_1, \alpha) - L(p, f_1, \alpha) + U(p, f_2, \alpha) - L(p, f_2, \alpha) \qquad (15)$$

$$\forall p \in P[a, b]$$

Since $f_1 \in R_\alpha$ and $f_2 \in R_\alpha$ over $[a, b]$.

Therefore every $\epsilon > 0$, there exist partition p_1, p_2 , such that

$$U(p_1, f_1, \alpha) - L(p_1, f_1, \alpha) < \epsilon/_2 \; ,$$

and $$U(p_2, f_2, \alpha) - L \, (p_2, f_2, \alpha) < \epsilon/_2 \; .$$

Let $p = p_1 \cup p_2$, then

$$U(p, f_1, \alpha) - L(p, f_1, \alpha) < \epsilon/_2 \; ,$$

and $U(p, f_2, \alpha) - L(p, f_2, \alpha) < \epsilon/_2$

This gives $U(p, f_1, \alpha) - L(p, f_1, \alpha) + U(p, f_2, \alpha) - L(p, f_2, \alpha) < \epsilon$.

Therefore (15) gives $U(p, f, \alpha) - L\ (p, f, \alpha) < \epsilon$.

Thus ∃ partition p of [a, b], such that

$$U(p, f, \alpha) - L\ (p, f, \alpha) < \epsilon\ .$$

Therefore $f \in R_\alpha[a, b]$.

Now we have to prove that

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.$$

Since the upper integral is the infimum of upper sums, therefore ∃ partition

p_1, p_2 , such that

$$U(p, f_1, \alpha) < \int_a^b f_1\, d\alpha + \frac{\varepsilon}{2}$$

and $$U(p, f_2, \alpha) < \int_a^b f_2 d\alpha + \frac{\epsilon}{2}\ .$$

For such partition p,

$$\int_a^b f d\alpha \leq U(p, f, \alpha) \leq U(p, f_1, \alpha) + U(p, f_2, \alpha) \text{ by} \qquad (16)$$

$$\leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \epsilon\ .$$

$$\int_a^b f d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \epsilon \qquad \forall\, \epsilon > 0\ .$$

Letting $\epsilon \to 0$ we get

$$\int_a^b f d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \qquad (17)$$

Simillarly by considering lower integrals as supremum of lower sums , we get

$$\int_a^b f d\alpha \geq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \qquad (18)$$

From (17) and (18), we get

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \ .$$

Hence the theorem follows.

Theorem 2.7. If $f_1, f_2 \in R_\alpha[a,b]$, then $f = f_1 - f_2 \in R_\alpha[a,b]$, and

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha \ .$$

Proof. Let $p = \{a = x_0, x_1, x_2, \ldots\ldots\ldots\ldots.., x_n = b\}$ be any partition of $[a,b]$, *and* m_i', M_i', m_i'', M_i'' and m_i, M_i are the infimum and supremum of f_1, f_2 and *f* respectively.

Then clearly bounds of ($-f_2$) are $-M_i''$ and $-m_i''$

Therefore $m_i'' - M_i'' \leq m_i \leq M_i \leq M_i' - m_i''$.

Multiplying by $\Delta\alpha_i$, we get

$$\Delta\alpha_i\ m_i' + \Delta\alpha_i\ (-M_i'') \leq \Delta\alpha_i m_i \leq \Delta\alpha_i M_i \leq \Delta\alpha_i\ M_i' + (-m_i'')\ \Delta\alpha_i$$

for i= 1,2,…n.

This implies that

$$\sum_{i=1}^n \Delta\alpha_i\, m_i' - \sum_{i=1}^n \Delta\alpha_i\, (M_i'') \leq \sum_{i=1}^n \Delta\alpha_i m_i \leq$$

$$\sum_{i=1}^n \Delta\alpha_i M_i \leq \sum_{i=1}^n \Delta\alpha_i\, M_i' - \sum_{i=1}^n m_i''\, \Delta\alpha_i \ .$$

Therefore $L(p, f_1, \alpha) - U(p, f_2, \alpha) \leq L(p, f, \alpha) \leq U(p, f, \alpha) \leq$

$$U(p, f_1, \alpha) - L(p, f_2, \alpha) \qquad (19)$$

$$U(p, f, \alpha) - L(p, f, \alpha) \leq U(p, f_1, \alpha) - L(p, f_2, \alpha) - L(p, f_1, \alpha) + U(p, f_2, \alpha).$$

This gives $U(p, f, \alpha) - L(p, f, \alpha) \leq (p, f_1, \alpha) - L(p, f_1, \alpha) +$

$$U(p, f_2, \alpha) - (p, f_2, \alpha) \qquad (20)$$

Let $\epsilon > 0$, *then* $\exists$ *partitions* p_1 and p_2 of $[a, b]$, such that

$$U(p_1, f_1, \alpha) - L(p_1, f_1, \alpha) \quad < \epsilon/_2 \quad ,$$

and

$$U(p_2, f_2, \alpha) - L(p_2, f_2, \alpha) \quad < \epsilon/_2 \ .$$

Let $\qquad p = p_1 \cup p_2$, then

$$U(p, f_1, \alpha) - L(p, f_1, \alpha) \quad < \epsilon/_2 \ ,$$

and

$$U(p, f_2, \alpha) - L(p, f_2, \alpha) \quad < \epsilon/_2 \ .$$

This gives

$$U(p, f_1, \alpha) - L(p, f_1, \alpha) + U(p, f_2, \alpha) - L(p, f_2, \alpha) < \in \ .$$

Therefore

$$U(p, f, \alpha) - L(p, f, \alpha) \quad < \in.$$

This shows that $f \epsilon R_\alpha[a, b]$.

Now we will show that

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha \ - \int_a^b f_2 d\alpha.$$

Since upper integrals and lower integrals are infimum and suprimum .

Therefore For any $\epsilon > 0$, *there* exist partition p_1 and p_2 , such that

$$\int_a^b f_1 d\alpha > U(p_1, f_1, \alpha) - \frac{\epsilon}{2} \ .$$

and $\qquad \int_a^b f_2 d\alpha \ < U(p_2, f_2, \alpha) - \frac{\epsilon}{2}$.

If $\qquad P = p_1 \cup p_2$, then

$$U(p, f_1, \alpha) < \int_a^b f_1 d\alpha + \frac{\epsilon}{2}.$$

and $$L(p, f_2, \alpha) > \int_a^b f_2 d\alpha - \frac{\epsilon}{2} \qquad (21)$$

For such partition *P*, we have

$$\int_a^b f d\alpha \le U(p, f, \alpha) \le U(p, f_1, \alpha) - L(p, f_2, \alpha) \qquad \text{by (19)}$$

$$< \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha + \epsilon \qquad \text{by (21)}.$$

This implies

$$\int_a^b f d\alpha < \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha + \epsilon .$$

letting $\epsilon \to 0$, we get $$\int_a^b f d\alpha \le \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha \qquad (22)$$

Proceeding with $(-f_1)$ and $(-f_2)$ in place of f_1, f_2 , we get

$$-\int_a^b f d\alpha \le -\int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

This implies that

$$\int_a^b f d\alpha \ \ge \ \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha$$

This gives

$$\int_a^b f d\alpha \ \ge \ \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha \qquad (23)$$

From (22) and (23), we get

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha.$$

Theorem 2.8. If $f \in R_\alpha[a,b]$, then $f \in R_\alpha[a,b]$ and $f \in R_\alpha[c,b]$ $\forall\ c \in [a,b]$ and conversely. Also in either case

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha .$$

Proof. Suppose that $f \in R_\alpha[a,b]$.

Therefore for any $\varepsilon > 0$, $\exists\, P$ of $[a, b]$, such that

$$U(p, f, \alpha) - L(p, f, \alpha) < \varepsilon.$$

Let $p* = P \cup \{c\}$

Then, $U(p*, f, \alpha) - L(p*, f, \alpha) \leq U(p, f, \alpha) - L(p, f, \alpha) < \varepsilon$.

Let p_1 and p_2 be the set of points of $p*$ between [a, c] and [c, b] respectively. Then

$$U(p*, f, \alpha) = U(p_1, f, \alpha) + U(p_2, f, \alpha) ,$$

and $L(p*, f, \alpha) = L(p_1, f, \alpha) + L(p_2, f, \alpha)$.

Also $U(p_1, f, \alpha) - L(p_1, f, \alpha) \geq 0,$

and $U(p_2, f, \alpha) - L(p_2, f, \alpha) \geq 0.$

We have

$$U(p_1, f, \alpha) + U(p_2, f, \alpha) - L(p_1, f, \alpha) - L(p_2, f, \alpha)$$

$$= U(p*, f, \alpha) - L(p*, f, \alpha) < \varepsilon$$

Therefore,

$$[U(p_1, f, \alpha) - L(p_1, f, \alpha)] + [U(p_2, f, \alpha) - L(p_2, f, \alpha)] < \varepsilon$$

Since each bracket on L.H.S is non negative .

Therefore, $U(p_1, f, \alpha) - L(p_1, f, \alpha) < \frac{\epsilon}{2}$,

and $U(p_2, f, \alpha) - L(p_2, f, \alpha)] < \frac{\epsilon}{2}$

where $p_1 \in P[a, c]$ and $p_2 \in P[c, b]$.

Therefore, $f \in R_\alpha\,[a, c]$ and $f \in R_\alpha\,[c, b]$; $a \leq c \leq b.$

Conversely, suppose $f \in R_\alpha\,[a, c]$ and $f \in R_\alpha\,[c, b]$; $a \leq c \leq b.$

Therefore for $\epsilon > 0$, we can find partitions p_1, p_2 of $[a, b], [a. c]$ respectively, such that

$$U(p_1, f, \alpha) - L(p_1, f, \alpha) < \frac{\epsilon}{2}$$

and $$U(p_2, f, \alpha) - L(p_2, f, \alpha) < \frac{\epsilon}{2}.$$

Let $p^* = p_1 \cup p_2$, then clearly p* is a partition of [a, b].

Also, $U(p^*, f, \alpha) - L(p^*, f, \alpha) = [U(p_1, f, \alpha) + U(p_2, f, \alpha)] -$

$[L(p_1, f, \alpha) + L(p_2, f, \alpha)]$

$= U(p_1, f, \alpha) - L(p_1, f, \alpha) + U(p_2, f, \alpha) - (p_2, f, \alpha)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Thus $\exists$ partition $p*$ of $[a, b]$, such that

$$U(p*, f, \alpha) - L(p*, f, \alpha) < \epsilon$$

Therefore, $f \in R_\alpha[a, b]$.

We know that for any two function f_1 and f_2.

If $f = f_1 + f_2$, then

$$inf\ f \geq inf\ f_1 + inf\ f_2,$$

and $$Sup\ f \leq sup\ f_1 + sup\ f_2.$$

Now for any partition p_1, p_2 of [a,c],[c, b] respectively, if $p^* = p_1 \cup p_2$, then

$$U(p*, f, \alpha) = U(p_1, f, \alpha) + U(p_2, f, \alpha)$$

$\therefore$ $$Inf\{U(p*, f, \alpha)\} \geq inf\{U(p_1, f, \alpha)\} + inf\{U(p_2, f, \alpha)\}$$

Therefore $$\int_a^{-b} f d\alpha \geq \int_a^{-c} f d\alpha + \int_c^{-b} f d\alpha$$

$$\int_a^b f d\alpha \geq \int_a^c f d\alpha + \int_c^b f d\alpha \qquad (24)$$

Preceeding with $(-f)$in place of f, we get

$$\int_a^b f d\alpha \leq \int_a^c f d\alpha + \int_c^b f d\alpha \tag{25}$$

From (24) and (25), we get

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha \ .$$

Theorem 2.9. The oscillation of a bounded function *f* on an interval [*a,* b] is supremum of the set $\{|f(x_1) - (x_2)| \ , \ x_1, x_2 \in [a,b]\}$ of numbers.

Proof. Let *M, m* be the bounds of *f* on $[a,b]$.

Now $m \leq f(x_1), \ f(x_2) \leq M; \quad \forall \ x_1, x_2 \in [a,b]$.

This implies that

$$|f(x_1) - f(x_1)| \ \leq M - m \qquad \forall \ x_1 \, , \, x_2 \in [a,b].$$

Therefore, $M - m$ is an upper bound of the set $\{ \ |f(x_1) - f(x_2)| : \ x_1, x_2 \in [a,b]\}$.

Let $\epsilon > 0$ *be any number*.Since M is supremum of *f*, therefore there exists $x' \in [a,b]$, such that

$$f(x') > M - \tfrac{\epsilon}{2} \ .$$

Simillarly $\exists \ x'' \ \in [a,b]$, such that

$$f(x'') < m + \tfrac{\epsilon}{2} \quad .$$

This gives $f(x') - f(x'') \ > \ M - m - \epsilon$.

$\Rightarrow \qquad |f(x') - f(x'')| > M - m - \epsilon \cdot$

Thus , $\exists$ a no. in the set $\{|f(x') - f(x'')| : \ x_1, x_2 \in [a,b]\}$, such that

$$|f(x') - f(x'')| > M - m - \epsilon \ \ \forall \ \epsilon > 0 \ .$$

This shows that $M - m - \ \epsilon$ is not upper bound of the above set $\forall \ \epsilon > 0$.

This gives $M - m$ is *l.u.b*. of the above set.

That is, $M - m = sup\{|f(x_1) - f(x_2)|: x_1, x_2 \in [a, b]\}$.

Theorem 2.10. If $f_1, f_2 \in R_\alpha[a, b]$, then $f_1 f_2 \in R_\alpha[a, b]$.

Proof. Since f_1 and f_2 are bounded on [*a,* b], therefore $f_1 f_2$ is also bounded on [*a,* b].

Let $p = \{a = x_0, x_1, x_2, \dots\dots\dots\dots.., x_n = b\}$ be any partition of [*a, b*].

Suppose m_i', M_i', m_i'', M_i'' and m_i, M_i are the infimum and supremum of f_1, f_2 and *f* respectively in Δx_i .We have for all $x_1, x_2 \in \Delta x_i$

$$f_1 f_2 (x_2) - f_1 f_2 (x_1) = f_1 (x_2) f_2 (x_2) - f_1 (x_1) f_2 (x_1)$$

$$= f_1(x_2) f_2 (x_2) - f_1 (x_1) f_2 (x_1) + f_1 (x_1) f_2 (x_2) - f_1 (x_1) f_2 (x_1)$$

$$= f_2(x_2)(f_1(x_2) -- f_1(x_1)) + f_1 (x_1) (f_2 (x_2) - f_2 (x_1)).$$

This implies $|(f_1f_2)(x_2) - (f_1f_2)(x_2)| \leq |f_2(x_2)||f_1(x_2) - f_1(x_1)| +$

$$|f_1(x_1)||f_2(x_2) - f_2(x_1)|$$

$$\leq K[M_i' - m_i'] + K[M_i - m_i]$$

where $|f_1(x)| \leq K \ and \ |f_2(x)| \leq K \quad \forall x \in [a, b]$.

This gives

$$M_i - m_i \leq K[M_i' - m_i'] + K[M_i'' - m_i''] \tag{26}$$

Now let $\epsilon > 0$ be any number.

Since f_1, f_2 are integrable, therefore $\exists$ partitions p_1, p_2, such that

$$U(p_1, f_1, \alpha) - L(p_1, f_1, \alpha) < \frac{\epsilon}{2k} \tag{27}$$

and $$U(p_2, f_2, \alpha) - L(p_2, f_2, \alpha) < \frac{\epsilon}{2k} \tag{28}$$

Let $P = p_1 \cup p_2$, then

$$U(p, f_1, \alpha) - L(p, f_1, \alpha) < \epsilon/2k,$$

and

$$U(p, f_2, \alpha) - L(p, f_2, \alpha) < \epsilon/2k.$$

For this partition, we must have

$$M_i\text{-}m_i \le k[M_i' - m_i'] + k[M_i'' - m_i''] \quad \text{by (26)}$$

This gives

$$U(p, f_1 f_2, \alpha) - L(p, f_1 f_2, \alpha) \le$$

$$k[U(p, f_1, \alpha) - L(p, f_1, \alpha)] + k[U(p, f_2, \alpha) - L(p, f_2, \alpha)]$$

$$< k\frac{\epsilon}{2k} + k\frac{\epsilon}{2k} \quad \text{by (27) and (28).}$$

This gives

$$U(p, f, \alpha) - L(p, f, \alpha) < \epsilon \ .$$

This shows that $f = f_1 f_2$ is also integrable with respect to α over $[a, b]$.

Theorem 2.11. If f_1 and f_2 are two bounded and integrable functions with respect to α over $[a, b]$ and $\exists$ a number $\lambda > 0$, such that $|f_2(x)| \ge \lambda, \forall x \in [a, b]$, then $\frac{f_1}{f_2}$ is also integrable with respect to α over $[a, b]$.

Proof. Since f_1, f_2 are bounded and $|f_2(x)| \ge \lambda$, $\forall x \in [a, b]$.

Also $\exists\ K$, such that $|f_1(x)| \le K, \forall\ x \in [a, b]$.

Therefore, $\left|\frac{f_1(x)}{f_2(x)}\right| = \left|\frac{f_1(x)}{f_2(x)}\right| \le \frac{K}{\lambda}$; $\forall\ x \in [a, b]$.

Thus, f_1/f_2 is bounded function on $[a, b]$.

Let $\epsilon > 0$ be any number, therefore $\exists$ partitions p_1, p_2 of $[a, b]$, such that

$$U(p_1, f_1, \alpha) - L(p_1, f_1, \alpha) < \epsilon\lambda^2/2K.$$

and $U(p_2, f_2, \alpha) - L(p_2, f_2, \alpha) < \epsilon\lambda^2/2K$.

Let P = $p_1 \cup p_2$, then

$$U(p, f_1, \alpha) - L(p, f_1, \alpha) < \epsilon\lambda^2/2K \qquad (29)$$

and $$U(p, f_2, \alpha) - L(p, f_2, \alpha) < \epsilon\lambda^2/2K \qquad (30)$$

Let p = $\{a = x_0, x_1, x_2 \ldots x_n = b\}$ be any partition of $[a, b]$.

Suppose m_i', M_i', m_i'' , M_i'' and m_i, M_i are the infimum and supremum of f_1, f_2 and f respectively,we have

$$\left|\left(\frac{f_1}{f_2}\right)(x_2) - \left(\frac{f_1}{f_2}\right)(x_1)\right| = \left|\frac{f_1(x_2)}{f_2(x_2)} - \frac{f_1(x_1)}{f_2(x_1)}\right|$$

$$= \left|\frac{f_1(x_2)f_2(x_1) - f_1(x_1)f_2(x_2)}{f_2(x_2).f_2(x_1)}\right|$$

$$\leq \frac{1}{\lambda^2}\ [|f_1(x_2)f_2(x_1) - f_1(x_1)f_2(x_2)|].$$

This implies that

$$\left|\left(\frac{f_1}{f_2}\right)(x_2) - \left(\frac{f_1}{f_2}\right)(x_1)\right| \leq \frac{1}{\lambda^2}\left[\left|\begin{matrix} f_1(x_2)f_2(x_1) - f_2(x_1)f_1(x_1) + \\ f_2(x_1)f_1(x_1) - f_1(x_1)f_2(x_2) \end{matrix}\right|\right]$$

$$\leq \frac{1}{\lambda^2}[|f_1(x_1)||f_1(x_2) - f_1(x_1)| + |f_1(x_1)||f_2(x_1) - f_2(x_2)|]$$

$$\leq \frac{1}{\lambda^2}[K\{M_i' - m_i'\} + \{M_i''\text{-}m_i''\}]$$

$$= \frac{K}{\lambda^2}\left[\{M_i' - m_i'\} + \{M_i''\text{-}m_i''\}\right].$$

This implies $$M_i - m_i \leq \frac{K}{\lambda^2}\left[\{M_i' - m_i'\} + \{M_i''\text{-}m_i''\}\right]$$

This gives

$$U\left(p, \frac{f_1}{f_2}, \alpha\right) - L\left(p, \frac{f_1}{f_2}, \alpha\right) \leq \frac{K}{\lambda^2}[U(p, f_1, \alpha) + L(p, f_1, \alpha) + U(p, f_2, \alpha) + L(p, f_2, \alpha)]$$

$$< \frac{K}{\lambda^2}\left[\frac{\lambda^2\epsilon}{2K} + \frac{\lambda^2\epsilon}{2K}\right] = \epsilon \qquad \text{by (29)and (30).}$$

Therefore, $U\,(p, \frac{f_1}{f_2}, \alpha) - L\left(p, \frac{f_1}{f_2}, \alpha\right) < \epsilon$.

Hence $\frac{f1}{f2} \in R_\alpha[a, b]$.

Theorem 2.12. If $f \in R_\alpha[a, b]$, then $|f| \in R_\alpha[a, b]$ and

$$\left|\int_a^b f d\alpha\right| \leq \int_a^b |f|\, d\alpha$$

Proof. We know that if $f \leq g$ on $[a, b]$, then

$$\int_a^b f d\alpha \leq \int_a^b g d\alpha \qquad (31)$$

Since f is bounded , therefore $\exists\, K > 0$, *such that*

$$|f(x)| \leq K, \quad \forall\, x \in [a, b].$$

Therefore the function $|f|$ is also bounded.

Again, since *f* is integrable, therefore for any $\epsilon > 0$, $\exists$ partition P of $[a, b]$, such that

$$U(p, f, \alpha) - L(p, f, \alpha) < \epsilon \qquad (32)$$

Let M_i' m_i' and M_i ,m_i be the bounds of *f* and $|f|$ in Δx_i ,we have $\forall\, x_1, x_2 \in \Delta x_i$

$$\big||f(x_2)| - |f(x_1)|\big| \leq |f(x_2) - f(x_1)| \leq M_i' - m_i' .$$

This implies that $M_i' - m_i' \leq M'_i - m'_i$

This implies for any partition *P* of [*a, b*];

$$U\,(p, |f|, \alpha) - L\,(p, |f|, \alpha) \leq U(p, f, \alpha) - L(p, f, \alpha)$$

$$< \epsilon \quad \text{by (32).}$$

Hence $|f| \in R_\alpha[a, b]$.

Also, $f(x) \leq |f(x)|$ *and* $-f(x) \leq |f(x)| \quad \forall\, x \in [a, b]$.

Therefore by (1), we have

$$\int_a^b f(x)d\alpha \leq \int_a^b |f|\, d\alpha$$

and $\quad -\int_a^b f d\alpha \leq \int_a^b |f|\, d\alpha$

This gives $\quad \left|\int_a^b f d\alpha\right| \leq \int_a^b |f|\, d\alpha.$

Theorem 2.13. If $f \in R_\alpha[a, b]$, then $f^2 \in R_\alpha[a, b]$.

Proof. Since f is bounded on $[a, b]$ therefore $\exists$, a numder $K > 0$, *such that*

$$|f(x)| \leq K \quad \forall\, x \in [a, b].$$

We have $\quad |f^2(x)| = |f(x)||f(x)| \leq K^2 \quad \forall\, x \in [a, b].$

This shows that f^2 is also bounded.

Let $M_i'\ m_i'$ be bounds of $|f|$ and $M_i'\ m_i'$, be those of f^2 in Δx_i,

Then, $\quad M_i = M_i'^2 \quad m_i = m_i'^2 \qquad (33)$

Also, since $f \in R_\alpha[a, b]$, then for any $\epsilon > 0$, $\exists$ *a partition P of* $[a, b]$, such that

$$U(p, f, \alpha) - L(p, f, \alpha) < \frac{\epsilon}{2k}$$

Now, we have

$$\begin{aligned} U(p, f^2, \alpha) - L(p, f^2, \alpha) &= \sum_{i=1}^n (M_i - m_i)\, \Delta\alpha_i \\ &= \sum_{i=1}^n (M_i'^2 - m_i'^2)\, \Delta\alpha_i \\ &= \sum_{i=1}^n (M_i' + m_i')(M_i' - m_i')\, \Delta\alpha_i \\ &\leq \sum_{i=1}^n (K + K)(M_i' - m_i')\, \Delta\alpha_i \\ &= 2K \sum_{i=1}^n (M_i' - m_i')\Delta\alpha_i \\ &= 2K\, [U(p, f, \alpha) - L(p, f, \alpha)] < 2K.\frac{\epsilon}{2K} = \epsilon\,. \end{aligned}$$

Thus $\exists$ partition P of $[a, b]$, such that

$$U(p, f^2, \alpha) - L(p, f^2, \alpha) < \epsilon \quad \forall\, x > 0$$

Hence $f^2 \in R_\alpha [a, b]$.

The integral as a limit of sum.

Earlier, we arrived at the integral of functions via the upper and lower sums. The number M_i, m_i which appear in these sums are not necessary the values of the functions (they are the values of *f* if f is continuous). We shall now that $\int f d\alpha$ can also be considered as a limit of a sequence of sums in which M_i and m_i are replaced by the values of *f*.

Definition 2.3. Corresponding to partition p of [*a, b*] and $t_i \in \Delta x_i$,

Consider the sum $S\,(p, f, \alpha) = \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i$.

We say that *S (p, f,α)* converges to A as $\mu(p) \to 0$, that is

$$\lim_{\mu(p)\to 0} S(p, f, \alpha) = A,$$

If for every $\epsilon > 0$, *there exist* $\delta > 0$, *such that*

$$|s(p, f, \alpha) - A| < \epsilon, \quad \text{for every partition}$$

$P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ of $[a, b]$ with norm $\mu(p) < \delta$ *and* every choice of $t_i \in \Delta x_i$.

Theorem2.14 . If $\lim S(p, f, \alpha)$ exists as $\mu(p) \to 0$, *then*

$$f \in R_\alpha\,[a, b]$$

and
$$\lim_{\mu(p)\to 0} S(p, f, \alpha) = \int_a^b f(x)\,d\alpha.$$

Proof. Let us suppose that $\lim S(p, f, \alpha)$ exists as $\mu(p) \to 0$ and is equal to *A*.

Therefore for every $\epsilon > 0$, $\exists\ \delta > 0$ *such that for every partition P of* [*a, b*] with mesh $\mu(p) < \delta$ *and every choice of* t_i *in* Δx_i, we have

$$|S(p, f, \alpha) - A| < \frac{\epsilon}{2}$$

or $A - \frac{\epsilon}{2} < S(p,f,\alpha) < A + \frac{\epsilon}{2}$ (34)

Let *p* be one such partition. If we let the points t_i range over interval Δx_i and take the infimum and the supremum of the sums $S(p,f,\alpha)$. Therefore yields (34)

$$A - \frac{\epsilon}{2} < L(p,f,\alpha) \leq U(p,f,\alpha) < A + \frac{\epsilon}{2}$$

Therefore , $U(p,f,\alpha) - L(p,f,\alpha) < \epsilon$

Hence $\lim_{\mu\to 0} s(p,f,\alpha) = \int_a^b f(x)d\alpha = A.$

Theorem 2.15. If *f* is continuous on [*a, b*], then f ∈ R_α [*a, b*].

Moreover to every $\epsilon > 0, \exists$ a $\delta > 0$, *such that*

$$\left|S(p,f,\alpha) - \int_a^b f(x)d\alpha\right| < \epsilon \ .$$

For every partition $p = \{a = x_0, x_1, x_2, \ldots.. x_n = b\}$ of [*a, b*] with $\mu(p) < \delta$ *and for every choice of* t_i in Δx_i.

That is,

$$\lim_{\mu\to 0} s(p,f,\alpha) = \int_b^b f(x)d\alpha.$$

Proof. Let $\epsilon > 0$ *be given and let us choose* $\eta > 0$, *such that*

$$\eta[\alpha(b) - \alpha(a)] < \epsilon \qquad (35)$$

Since continuity of *f* on the [*a, b*] implies its uniform continuity on [*a, b*] . Therefore for $\eta > 0$ *there corresponding* $\delta > 0,$ *such that*

$$|f(t_1) - f(t_2)| < \eta, if\ |t_1 - t_2| < \delta, t_1, t_2 \in [a,b]. \qquad (36)$$

Let P be partition, with $\mu(p) < \delta$, *then in view of* (36)

$$M_i - m_i \leq \eta, \qquad i = 1,2, \ldots. n$$

$$\Rightarrow \quad U(p,f,\alpha) - L(p,f,\alpha) \leq \eta[\alpha(b) - \alpha(a)] < \epsilon \ .$$

Therefore, $f \in R_\alpha$ [a, b].

Again, if $f \in R_\alpha$ [a, b], then for every $\epsilon > 0$, $\exists$ $\delta > 0$, such that for all partitions P with $\mu(p) < \delta$

$$|S(p, f, \alpha) - L(p, f, \alpha)| < \epsilon .$$

Since $S(p, f, \alpha)$ and $\int_a^b f d\alpha$. both lie between $U(p,f,\alpha)$ and $L(p, f,\alpha)$ for all partitions p with $\mu(p) < \delta$ *and for all position of* t_i in $\Delta\alpha_i$.

$$\therefore \qquad \left|S(p, f, \alpha) - \int_a^b f(x) d\alpha\right| < U(p,f,\alpha) - L(p, f, \alpha) < \epsilon .$$

Hence $$\lim_{\mu \to 0} S(p, f, \alpha) = \int_a^b f d\alpha.$$

Theorem 2.16. If f is monotonic on [a, b] and if ɑ is contineous on $[a, b]$, then $f \in R_\alpha$ [a, b].

Proof. Let $\epsilon > 0$ be given positive number for any positive integer n, choose a partition $p = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ of $[a, b]$, such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}, \qquad i=1, 2, 3\ldots n.$$

This is possible because α is contineous and monotonic increasing on closed interval [a, b] and thus assumes every value between its bounded $\alpha(a)$ and $\alpha(b)$.

Let f be monotonic increasing on $[a, b]$, so that its lower and upper bounds m_i, M_i in Δx_i are given by

$$m_i = f(x_{i-1}), \quad M_i = f(x_i), \quad i = 1,2,3 \ldots n$$

Therefore,

$$U(p, f, \alpha) - L(p, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i)$$

$$= \frac{\alpha(b)-\alpha(a)}{n} \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b)-\alpha(a)}{n}(f(b) - f(a))$$

$$< \epsilon, \quad \text{for large } n.$$

Hence, $f \in R_\alpha [a, b]$.

Theorem 2. 17. If $f \in R_\alpha [a, b]$, and α is monotonic increasing on $[a, b]$, such that $\alpha' \in R [a, b]$, then $f \in R_\alpha [a, b]$ and

$$\int_b^b f d\alpha = \int_b^b f\alpha' dx.$$

Proof. Let $\epsilon > 0$ be any given number.

Since f is bounded, there exist M, such that

$$|f(x)| \leq M, \ \forall x \in [a, b] .$$

Again since $f, \alpha' \in R[a, b]$, therefore $f\alpha' \in R[a, b\}$and consequently $\exists$,

$\delta_1 > 0, \delta_2 > 0$, *such that*

$$|\sum f(t_i) \alpha'(t_i) \Delta x_i - \int f\alpha' dx| < \frac{\epsilon}{2} \tag{37}$$

for $\mu(p) < \delta_1$, *and all* t_i in Δx_i and

$$|\sum \alpha'(t_i)\Delta x_i - \int \alpha' dx| < \frac{\epsilon}{4m} \tag{38}$$

for $\mu(p) < \delta_2$, *and and* $t_i \in \Delta x_i$.

Now for $\mu(p) < \delta_2$ *and all* $t_i \in \Delta x_i$, $s_i \in \Delta x_i$, (38) gives

$$\sum|\alpha'(t_i) - \alpha'(s_i)| \Delta x_i < 2\frac{\epsilon}{4M} = \frac{\epsilon}{2M} \tag{39}$$

Let $\delta = min(\delta_1, \delta_2)$ and P any partition with $\mu(p) < \delta$.

Also by Lagranges Mean Value theorem , there are points $s_i \in \Delta x_i$, such that

$$\Delta\alpha i = \alpha'(s_i) \in \Delta x_i \tag{40}$$

Now for all $t_i \in \Delta x_i$;

$$|\sum f(t_i)\Delta\alpha_i - \int f\alpha' dx| = |\sum f(t_i)\alpha'(s_i)\Delta x_i - \int f\alpha' dx|$$

$$= |\sum f(t_i)\alpha'(t_i)\Delta x_i - \int f\alpha' dx + \sum f(t_i)[\alpha'(s_i) - \alpha'(t_i)]\Delta x_i|$$

$$\leq |\sum f(t_i)\alpha'(t_i)\Delta x_i - \int f\alpha' dx| + \sum |f(t_i)|\,|\alpha'(s_i) - \alpha'(t_i)\Delta x_i|$$

$$< \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon \qquad (41)$$

Hence for any $\epsilon > 0$, $\exists\ \delta > 0$, *such that for all partitions with* $\mu(p) < \delta$, (41) holds

Therefore, $\lim\limits_{\mu\to 0} \sum f(t_i)\Delta\alpha_i$ exists and equals $\int f\alpha' dx$.

Hence f $\in R_\alpha$ [*a, b*].

Therefore, $\int_b^b f d\alpha = \int_b^b f\alpha' dx.$

Theorem 2.18. (Particular case).

If *f* is continuous on [*a, b*] and α has a continuous derivative on $[a, b]$, then

$$\int_b^b f d\alpha = \int_b^b f\alpha' dx.$$

Proof. Under the given condition all the integral exists.

Let $p = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be any partition of [*a* ,*b*] .

Thus by Lagrange's Mean Value Theorem it is possible to find

$t_i \in]x_{i-1}, x_i[$, such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(xi - x_{i-1}) \qquad i=1,2,\ldots..n\ ,$$

or $\Delta\alpha_i = \alpha'(t_i)\Delta\alpha_i$.

Therefore

$$S(p, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} f(t_i)\ \alpha'(t_i)\Delta x_i$$
$$= S(p, f, \alpha').$$

Now letting $\mu(p) \to 0$, we get

$$\int_b^b f d\alpha = \int_b^b f\alpha' dx.$$

Theorem 2.19 . (First Mean Value Theorem).

If a function is continuous on [*a, b*] and α is monotonic increasing on $[a, b]$, then there exists a number $\xi \in [a, b]$, such that

$$\int_b^b f d\alpha = f(\xi)[\alpha(b) - \alpha(a)].$$

Proof: Since f is continuous function and α is monotonic function.

Therefore, $f \in R_\alpha$ [*a, b*].

Let *m* and *M* be infimum and supremum of *f* in [*a, b*], then

$$m[\alpha(b) - \alpha(a)] \leq \int_b^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$$

This implies that

$$m \leq \frac{\int_b^b f d\alpha}{\alpha(b) - \alpha(a)} \leq M$$

Let $$C = \frac{\int_b^b f d\alpha}{\alpha(b) - \alpha(a)}$$

Then, $\int_b^b f d\alpha = C\{\alpha(b) - \alpha(a)]$ where $m \leq C \leq M$.

Since *f* is continuous function therefore $\exists\ \xi \in [a, b]$.

$$f(\xi) = C.$$

Thus $\int_b^b f d\alpha = f(\xi)[\alpha(b) - \alpha(a)]$.

Hence the theorem.

Theorem 2.20. If f is continuous and α monotonic on $[a, b]$, then

$$\int_a^b f d\alpha = [f(x)\alpha(x)]_a^b - \int_a^b \alpha df$$

Proof. Under the given condition all the integrals exist.

Let $p = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a partition of $[a, b]$.

Choose $t_1, t_2, t_3, \ldots\ldots t_n$, such that

$$x_{i-1} \le t_i \le x_i,$$

Let $t_0 = a,\ \ t_{n+1} = b,$ so that

$$t_{i-1} \le x_{i-1} \le t_i.$$

Clearly $Q = \{a = t_0, t_1, t_2, \ldots\ldots t_n, t_{n+1} = b\}$ is also a pattition of $[a, b]$.

Now

$S(p, f, \alpha) = \sum_{i=1}^n f(t_i)\Delta\alpha_i$

$= f(t_1)[\alpha(x_1) - \alpha(x_0)] + f(t_2)[\alpha(x_2) - \alpha(x_1)] + \cdots + f(t_n)[\alpha(x_n) - \alpha(x_{n-1})]$

$= -\alpha(x_0)f(t_1) - \alpha(x_1)[f(t_2) - f(t_1)] + \alpha(x_2)[f(t_3) - f(t_2)] + \cdots\ \alpha(x_{n-1}) +$

$[f(t_n) - f(t_{n-1}] + \alpha(x_n)f(t_n)]$

Adding and subtracting $\alpha(x_0)f(t_0) + \alpha(x_n)f(t_{n+1})$

$S(p, f, \alpha) = \alpha(x_n)f(t_{n+1}) - \alpha(x_0)f(t_0) - \sum_{t=0}^n \alpha(x_i)[f(t_{i+1}) - f(t_i)]$

$$= f(b)\ \alpha(b) - f(a)\alpha(a) - S(Q, \alpha, f) \qquad (42)$$

If $\mu(p) \to 0$, then $\mu(Q) \to 0$, so we get

$$Lim\ S(p, f.\alpha) = \int_a^b f d\alpha$$

and $Lim\ S(Q, \alpha, f) = \int_a^b \alpha df.$

Hence proceeding $\mu(p) \to 0$, we get from (42)

$$\int_a^b f d\alpha = f(b)\,\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df .$$

Therefore, $\int_a^b f d\alpha = [f(x)\alpha(x)]_a^b - \int_a^b \alpha df$

This completes the proof.

The above theorem is also known as second Mean Value Theorem.

Fundamental Theorem of Calculus.

Theorem 2.21: A function $f \in R[a,b]$ and there exists a function F such that $F' = f$ on [a, b], then

$$\int_a^b f dx = F(b) - F(a) .$$

Proof. Since the function $F'=f$ is bounded and *integrable* therefore for given $\epsilon > 0$, $\exists\ \delta > 0$, *such that for every partition*

$$P = \{a = x_0, x_1, \dots\dots..x_n\} \text{ with norm } \mu(p) < \delta,$$

$$\left.\begin{array}{l} \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f dx\right| \\ or \\ \lim\limits_{\mu(p)\to 0} \sum_{i=1}^{n} f(t_I)\,\Delta x_i = \int_a^b f dx \\ \text{for every choice of points } t_i \text{ in } \Delta x_i . \end{array}\right\} \qquad (43)$$

Since we have freedom in the selection of points t_i in Δx_i , we choose them in particular way as follows

By Lagrange's Mean Value Theorem, we have

$$F(x_i) - F(x_{i-1}) = F'(x_i)\Delta x_i, \quad i = 1,2,3 \dots\dots n$$
$$= f(t_i)\Delta x_i$$

This implies that

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})]$$

$$= \quad F(b) - F(a)$$

Therefore , $\lim_{\mu(p)\to 0} \sum_{i=1}^{n} f(t_i)\,\Delta x_i = F(b) - F(a)$.

Hence $\int_a^b f dx = F(b) - F(a)$.

Remark. It is this theorem that shows under certain conditions integration and differentiation are reverse processes.