Improper Integrals.

The concept of Riemann integrals as developed in previous chapter requires that the range of integration is finite and the integrand remains bounded on that domain. if either (or both) of these assumptions is not satisfied it is necessary to attach a new interpretation to the integral

Definition 3.1. In case the integrand f becomes infinite in the interval $a \leq x \leq b$,

That is f has points of infinite discontinuity (singular points) in [*a, b*] or the limits of integration *a* or *b* (or both becomes infinite, the symbol $\int_a^b f dx$ is called an improper(or infinite or generalised) integral.

Thus,

$$\int_1^\infty \frac{dx}{x^2}, \qquad \int_{-\infty}^\infty \frac{dx}{1+x^2}. \quad \int_0^1 \frac{dx}{x(1-x)} \,, \quad \int_{-1}^\infty \frac{dx}{x(1-x)}$$

are examples of improper integrals.

The integrals which are not improper are called proper integral , thus

$\int_0^1 \frac{sinx}{x} dx$ is a proper integral.

Integration of Unbounded Function with finite limits of integration.

Definition 3.2. Let a function *f be* defined in a interval [*a, b*] everywhere except possible at finite number of points.

(i) Convergence at left –end. Let *a* be the only points of infinite discontinuity of *f* so that according to assumption made in the last section, the integral

$\int_{a+\lambda}^b f dx$ exists $\forall \lambda, 0 < \lambda < b - a.$

$\textit{The improper integral } \int_a^b f dx$ is defined as the

$\lim_{\lambda \to 0^+} \int_{a+\lambda}^b f dx$ so that,

$\int_a^b f dx = \lim_{\lambda \to 0} \int_{a+\lambda}^b f dx$.

If this Limit exists and is finite, the improper integral $\int_a^b f dx$ is said to converge at (a) if otherwise, it is called divergent.

Note. For any c $, a < c < b$

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx.$$

Then,

$$\int_a^b f dx \text{ and } \int_a^c f dx$$

converges and diverges together and $\int_c^b f dx$ is proper.

(ii) Convergence at right-end. Let *b* be the only point of infinite discontinuity the improper integral is then defined by the relation

$$\int_a^b f dx = \lim_{\mu \to 0^+} \int_a^{b-\mu} f dx \qquad 0 < \mu < b - a \ .$$

If the limit exists, the improper integral is said to be convergent at $b.$ Otherwise is called divergent.

Note : For the same reason as above,

$\int_c^b f dx$ and $\int_a^b f dx$ converges and diverges together $\forall \, c, a < c < b.$

(iii) Convergence at both the end points. If the end points *a* and *b* are the only points of infinite discontinuity of *f* , then for any point *c, a < c < b,*

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

If both the integrals are convergent as by case (i) and (ii),then

$\int_a^b f dx$ is convergent , otherwise it is divergent. The improper integral is also defined as:

$$\int_a^b f dx = \lim_{\substack{\lambda \to 0+ \\ \mu \to 0+}} \int_{a+\lambda}^{b-\mu} f dx.$$

The improper integral exists if the limit exists.

(iv) Convergence at Interior points. If an interior point *c*, *a* < *c* < *b*, is the only point of infinite discontinuity of f, we get

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx \qquad (1)$$

the improper integral $\int_a^b f dx$ exists of the both integral on R.H.S of (1) are exists.

Example 3.1. Examine the convergence of:

(i) $\int_0^1 \frac{dx}{x^2}$ *(ii)* $\int_0^1 \frac{dx}{\sqrt{i-x}}$ *(iii)* $\int_6^2 \frac{dx}{2x-x^2}$.

(i) 0 is the point of infinite discontinuity of integrand [0, 1].

Thus,

$$\int_0^1 \frac{dx}{x^2} = \lim_{\lambda \to 0+} \int_\lambda^1 \frac{dx}{x^2}, \qquad o < \lambda < 1$$

$$= \lim_{\lambda \to 0+} \left(\frac{1}{\lambda} - 1\right) = \infty$$

Thus the proper integral is divergent.

(ii) Home Assignment.

Home Assignment

Comparison Tests for Convergence At '*a*' of $\int_a^b f dx$.

Theorem 3.1. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f dx$ at '*a*' where *f* is positive in [*a, b*]. This is , $\exists$ a psoitive number M, independent of λ , such that

$$\int_{a+\lambda}^b f dx < M, \quad 0 < \lambda < b - a .$$

Proof. We know that the improper integral $\int_a^b f dx$ converges at '*a*' if for 0

$$0 < \lambda < b - a, \quad \int_{a+\lambda}^b f dx \text{ tends to finite limit as } \lambda \to 0^+ .$$

Since *f* is positive in $[a + \lambda, b]$, the positive function of λ , $\int_{a+\lambda}^b f dx$ is monotonic incereasing as λ , decreases and will therefore tend to a finite limit iff it is bounded above , This is, $\exists$ a positive number M independent of λ, such that

$$\int_{a+\lambda}^b f dx < M, \quad 0 < \lambda < b - a.$$

Hence the theorem is proved.

Note. If no such number *M* exists, the monotonic increasing function $\int_{a+\lambda}^b f dx$ is not bounded above and therefore tend to $+\infty$ as $\lambda \to 0^+$, and hence the improper integral $\int_a^b f dx$ diverges to $+\infty$.

Comparison Test.

Theorem 3.2: If *f* and *g* are two positive functions and '*a*' is only singular point of *f* and *g* on [*a. b*], such that

$$f(x) \leq g(x), \text{ for all } x \in [a, b]$$

(i) $\int_a^b f dx$ converges, if $\int_a^b g dx$ converges .

(ii) $\int_a^b g dx$ diverges, if $\int_a^b f dx$ converges .

Proof. Since *f* and *g* are two positive functions on [*a, b*] and '*a*' is only singular point of *f* and *g*. Therefore f and g are bound in $[a+\lambda, b]$, for all $0 < \lambda < b-a$.

Also Since, *f(x)* $\leq g(x), for\ all\ x \in [a, b]$, implies

$$\int_{a+\lambda}^b f dx \leq \int_{a+\lambda}^b g dx \qquad \text{(i)}$$

(1) Suppose $\int_a^b g dx$ be convergent, so that $\exists\ m > 0$, $such\ that$ *for all* $\lambda, 0 < \lambda < b-a$,

$$\int_{a+\lambda}^b f dx < m .$$

From (i) we have

$$\int_{a+\lambda}^b f dx < m , \qquad \text{for all } \lambda , 0 < \lambda < b-a .$$

Hence $\int_a^b f dx$ is convergent.

(2) Now suppose $\int_a^b f dx$ is divergent then the positive function $\int_{a+\lambda}^b f dx$is not bounded above.

Therefore from (i) it follows that the positive function $\int_{a+\lambda}^b g dx$ is not bounded above.

Hence $\int_{a+\lambda}^b g dx$ is divergent. This completes the Theorem.

Comparison Test (limit form).

Theorem 3.3. If *f* and *g* are two positive functions [*a, b*] and '*a*' is the only singular point of *f* and *g* in [*a, b*], such that

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = l \qquad \text{where 'l' is a non – zero finite number.}$$

Then, the two integrals $\int_a^b f dx$ and $\int_a^b g dx$ converges and diverges together at '*a*'.

Proof. Evidently, $1 > 0$. Let ε be positive number such that $1 - \varepsilon > 0$.

Since, $\lim\limits_{x \to a^+} \frac{f(x)}{g(x)} = 1.$

Therefore there exists a nbd of] a, *c* [, *a* < *c* < *b* , such that for all $x \in]a, c[$

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon$$

or $$(l - \varepsilon)g(x) < f(x) < (l + \varepsilon)g(x) \text{ .}$$

This implies that

$$(l - \varepsilon)g(x) < f(x) \qquad (2)$$

and

$$f(x) < (l + \varepsilon)g(x) \qquad (3)$$

$$\forall\, x \in]a, c[$$

If $\int_a^b f dx$ converges, then from (i)

$\int_a^b g(x) dx$ also converges at a .

If $\int_a^b f dx$ diverges , then from (ii)

$\int_a^b f dx$ diverges at a .

If in the above Theorem, $\lim\limits_{x \to a^+} \frac{f(x)}{g(x)} \to 0$ *and* $\int_a^b g dx$ *converges*, then

$\int_a^b f dx$ *converges* and if

$\underset{x\to a^+}{Lim}\frac{f(x)}{g(x)} \to \infty$ and $\int_a^b g dx$ diverges, then, $\int_a^b f dx$ also diverges.

Useful Comparison Integral.

Theorem 3.4. The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ Converges if and only if *n < 1.*

Proof. It is proper integral if *n* ≤ 0 and improper for all other values of *n, 'a'* being only singular point of the integrand.

Now for *n* $\neq 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\lambda\to 0^+} \int_{a+\lambda}^b \frac{dx}{(x-a)^n} ,$$

$$= \lim_{\lambda\to 0^+} \frac{1}{-n+1}[(b-a)^{-n+1}-\lambda^{-n+1}]$$

$$= \begin{cases} \frac{1}{-n+1}[(b-a)^{-n+1}, & if\ n<1 \\ \infty & if\ n>1 . \end{cases}$$

Also for *n = 1*

$$\underset{\lambda\to 0^+}{Lim} \int_{a+\lambda}^b \frac{dx}{x-a} = \lim_{\lambda\to 0^+}[log(b-a) - log\lambda] = \infty.$$

Thus, $\int_a^b \frac{dx}{(x-a)^n}$ converges for *n < 1 .*

Note. A similar result holds for convergence of $\int_a^b \frac{dx}{(b-x)}$ $at\ b.$

Example 3.2. Test the convergence of

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ *(ii)* $\int_0^{\pi/2} \frac{sinx}{x^p}dx$

Solution. Let *f(x)* = $\frac{1}{\sqrt{1-x^3}}$

$$= \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$$

$$= \frac{1}{(1+x+x^2)^{\frac{1}{2}}} \cdot \frac{1}{(1-x)^{\frac{1}{2}}}$$

Clearly, $\frac{1}{(1+x+x^2)^{\frac{1}{2}}}$ is a bounded function.

Let M be its upper bound, then ,

$$\frac{1}{(1+x+x^2)^{\frac{1}{2}}} \cdot \frac{1}{(1-x)^{\frac{1}{2}}} \leq \frac{M}{(1-x)^{\frac{1}{2}}} \quad , \qquad x \in [1, 0].$$

Also since $\int_0^1 \frac{mdx}{(1-x)^{\frac{1}{2}}}$ is convergent as $n = \frac{1}{2} < 1.$

Therefore, $\frac{1}{\sqrt{1-x^3}}$ is convergent.

(i) For $p \leq 1$, it is a proper integral for $p > 1$, it is an improper integral 0 being the point of infinite discontinuity

Now $\frac{sinx}{x^p} = \frac{1}{x^{p-1}}\left(\frac{sinx}{x}\right)$

The function $\frac{sinx}{x}$ is bounded and $\frac{sinx}{x} \leq 1.$

Therefore, $\frac{sinx}{x^p} \leq \frac{1}{x^{p-1}}$

Also $\int_0^{\pi/2} \frac{dx}{x^{p-1}}$ converges only if $p-1 < 1$ or $p < 2$.

Therefore by comparison test $\int_0^{\frac{\pi}{2}} \frac{sinx}{x^p}$dx converges for *p* < *2* and diverges for $p \geq 2$.

Note. If $\lim_{x \to 0+} [(x-a)^n f(x)]$ exists and is non- zero finite, then, the integral $\int_a^b f dx$ convergs iff $n < 1$.

Example 3.3. Find the values of m and n for which the following integrals converges.

(i) $\int_0^1 e^{-mx}\, x^n dx$.

(ii) $\int_0^1 (\log \frac{1}{x})^m dx$.

Solution (i) Let k be positive number greater than 1,

Then, $e^{-mx}x^n \leq kx^n$, $\forall\ x \in [0, 1]$ and *m* ;

Also $\int_0^1 x^n = \int_0^1 \frac{dx}{x^{-n}}$ *converges for* $-n < 1$,

that is, *n > -1 only .*

Thus, $\int_0^1 e^{-mx}\, x^n dx$ converges only for *n >-1* and $\forall$ *m.*

(iii) Let(x) = $(\log \frac{1}{x})^m$ converges at *x= 0 and*

$\int_0^{\frac{1}{2}} (\log \frac{1}{x})^m\, dx$ is proper integral if $m \leq 0$. *Also*'0' is the only singular point if *m > 0*.

For *m > 0*,

Take $g(x) = \frac{1}{x^p}$, *0 < p < 1, s*o that

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} x^p (\log \frac{1}{x})^m$$

$$= 0, \text{ for } 0 < p < 1.$$

Therefore, $\int_0^{\frac{1}{2}} (\log \frac{1}{x})^m\, dx$ converges for all *m*.

Convergence at *x=1*

$\int_{\frac{1}{2}}^1 (\log \frac{1}{x})^m$ is proper integral for $m \geq 0$ and '1' is singular point , if m< 0.

For $m < 0$, take $g(x) = \frac{1}{(1-x)^{-m}}$, *s*o that $\lim_{x\to 1} \left[\frac{\log\frac{1}{x}}{1-x}\right]^m = 1.$

Since $\int_{\frac{1}{2}}^1 g dx$ converges for $-m < 1$ that is for $m > -1$.

Thus, $\int_0^1 (log\frac{1}{x})^m$dx converges for > -1 .

Hence $\int_0^1 (log\frac{1}{x})^m dx$ converges for *0* > $m > -1$*.*

Example 3.4. Show that (1) $\int_0^1 \frac{logx}{\sqrt{x}} dx$ is convergent.

(2) $\int_1^2 \frac{\sqrt{x}}{logx} dx$ is divergent.

Solution. (1) Since $\frac{logx}{\sqrt{x}}$ is negative on [*0, 1*].

Therefore we take *f(x)* = $-\frac{logx}{\sqrt{x}}$

$$= \frac{logx^{-1}}{\sqrt{x}} = \frac{log\, 1/x}{\sqrt{x}},$$

'0' is the only singular point.

Let

$$g(x) = \frac{1}{x^{\frac{3}{4}}} \quad , \qquad n = \frac{3}{4} < 1$$

We have

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} x^{\frac{1}{4}} log\frac{1}{x} = 0$$

Since $\int_0^1 g(x)dx$ converges.

Therefore, $\int_0^1 f(x)dx$ converges implies that $\int_0^1 \frac{logx}{\sqrt{x}} dx$ converges.

(2) Let *f(x)* = $\int_1^2 \frac{\sqrt{x}}{logx} dx$ *.*

Here *x=1* is only singular point.

Take $g(x) = \frac{1}{x-1}$, then

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{(x-1)\sqrt{x}}{logx}$$

$$= \lim_{x\to 1} \frac{x^{\frac{3}{2}} - x^{\frac{1}{2}}}{\log x}$$

$$= \lim_{x\to 1} \frac{\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{x}}$$

$$= \lim_{x\to 1} \frac{3}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{1}{2}}$$

$$= \frac{3}{2} - \frac{1}{2} = 1(\neq 0).$$

Thus, $\int_1^2 f dx$ and $\int_1^2 g dx$ behave same.

Since $\int_1^2 g dx$ is divergent.

Hence $\int_1^2 f dx$ is divergent.

Example 3.5. Show that $\int_0^{\frac{\pi}{2}} \left(\frac{\sin^m x}{x^n}\right) dx \;\; exists\ iff \;\; n < m + 1$

Solution. Let *f(x)* = $\left(\frac{\sin^m x}{x^n}\right)$

$$= \left(\frac{\sin x}{x}\right)^m . \frac{1}{x^{n-m}}$$

Here as $x \to o+$, $f(x) \to 0$ if $n - m < 0$, and *f (x)*→∞ if $n - m > 0$.

Thus it is proper integral if $n \leq m$ and improper if $n > m$.

'0' being the only point of infinite discontinuity.

When $m > n$,

Let $g(x) = \frac{1}{x^{n-m}}$, so that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^m = 1.$$

Also, $\int_0^{\frac{\pi}{2}} g dx = \int_0^{\frac{\pi}{2}} \frac{1}{x^{n-m}} dx$ converges, Iff $n-m < 1$.

That is, $n < m + 1$.

Therefore $\int_0^{\frac{\pi}{2}}\left(\frac{sin^m x}{x^n}\right)dx$ also converges iff $n < m+1$.

Example 3.6. (Beta Function). Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff m, n are both positive.

Proof. It is a proper integral for $m \geq 1, n \geq 1$, 0 and *1* are the only points of infinite discontinuity; *0* when *m* < *1* and *1.*

When n < *1,* we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx + \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$$

convergence at '*0*' , when *m* < *1.*

Let *f(x)* = $x^{m-1}(1-x)^{n-1}$

$$= \frac{(1-x)^{n-1}}{x^{1-m}} \quad .$$

Take $g(x) = \frac{1}{x^{1-m}}$,

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$.

Since $\int_0^{\frac{1}{2}} g dx$ converges *if and only if, 1-m < 1 or m> 0.*

Thus, $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ converges for *m* > *0*.

Convergence at *x=1* ,

When *n< 1*,

Let *f(x)* = $x^{m-1}(1-x)^{n-1}$

$$= \frac{(1-x)^{m-1}}{x^{1-n}}$$

Take $g(x) = \frac{1}{x^{1-n}}$, then

$$\lim_{x \to 1} \frac{f(x)}{g(x)} = 1.$$

Also, $\int_{\frac{1}{2}}^{1} g dx = \int_{\frac{1}{2}}^{1} \frac{1}{(1-x)^{1-n}} dx$ converges if and only if $1-n < 1$ or $n > 0.$

Thus, $\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1} dx$ converges if $n > 0$.

Hence $\int_{0}^{1} x^{m-1}(1-x)^{n-1} dx$ converges if $m > 0$, $n > 0$.

Example 3.7. For what values of m and n is the integral

$\int_{0}^{1} x^{m-1}(1-x)^{n-1} \log x \, dx$ convergent.

Solution . The integrand is negative in $[0, 1]$, therefore we shall test for the convergence of

$$\int_{0}^{1} -x^{m-1}(1-x)^{n-1} \log x dx$$

$$= \int_{0}^{1} x^{m-1}(1-x) \log \frac{1}{x} dx$$

Since 0 and 1 are only possible singular points of integrand. We have

$$\int_{0}^{1} x^{m-1}(1-x)^{n-1} \log \frac{1}{x} dx$$

$$= \int_{0}^{\frac{1}{2}} x^{m-1}(1-x)^{n-1} \log \frac{1}{x} dx + \int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1} \log \frac{1}{x} dx .$$

Convergence at 0.

It is proper integral for $m-1 > 0$ and improper for $m \leq 1$.'0' being the only point of infinite discontinuity.

Then, for $m \leq 1$

Let $f(x) = x^{m-1}(1-x)^{n-1} \log \frac{1}{x}$

$$= (1-x)^{n-1} \log \frac{\frac{1}{x}}{x^{1-m}}$$

Take $g(x) = \frac{1}{x^p}$

Also , $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} x^{p+m-1}(1-x)^{n-1} \log\frac{1}{x}$

$= 0$

If $p + m-1 >$ or $m > 1- p$.

Also $\int_0^{\frac{1}{2}} \frac{1}{x^p}\, dx$ *converges for* $1- p > 0$.

Thus

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1} \log\frac{1}{x} dx \text{ converges for } m > 1-p > 0.$$

converges at $x=1$

For $n < 0$,

Let $f(x) = x^{m-1}(1-x)^{n-1} \log\frac{1}{x}$

$$= \frac{x^{m-1}\log\frac{1}{x}}{(1-x)^{-n+1}}$$

Take $g(x) = \frac{1}{(1-x)^q}$.

Therefore $\int_{\frac{1}{2}}^{1} g(x)$ converges for $q-1 < 0$.

Also $\lim_{x\to 1_-} \frac{f(x)}{g(x)} = \lim_{x\to 1_-} \frac{x^{m-1}\log\frac{1}{x}}{(1-x)^{1-n-q}} = l$

where l is infinite if $1-n-q \leq 1$.

That is if $n \geq -q > -1$.

Thus, $\int_{\frac{1}{2}}^{1} f dx \text{ converges if } n > -1$.

Hence the given integral is convergent when *m > 0, n >-1.*

Example 3.8. Show that $\int_0^{\frac{\pi}{2}} \log sinx dx$ converges and also evaluate it.

Solution. Let *f(x) = log sinx* , then *f* is negative in [*0,π/2*] .

Therefore we consider –f instead of f.

Clearly '0' is only point of infinite discontinuity.

Let $g(x) = \frac{1}{x^m}$, *m < 1,*

Then,

$$\lim_{x \to 0+} \frac{-f(x)}{g(x)} = \lim_{x \to 0+} -x^m logsinx = 0,\ m < 1$$

Since $\int_0^{\frac{\pi}{2}} \frac{1}{x^m} dx$ converges for *m< 1,* thus

$\int_0^{\frac{\pi}{2}} \log sinx dx$ converges.

Let $I = \int_0^{\frac{\pi}{2}} \log sinx dx$.

We know that, *sin2x = 2sinxcosx.*

Therefore, *logsin2x = log2 + logsinx + logcosx.*

This implies that

$$\int_0^{\frac{\pi}{2}} \log sin2x dx = \int_0^{\frac{\pi}{2}} \log 2 dx + \int_0^{\frac{\pi}{2}} \log sinx dx + \int_0^{\frac{\pi}{2}} \log cosx dx$$

$$= \frac{\pi}{2} log2 + I + \int_0^{\frac{\pi}{2}} \log cosx dx$$

Put *2x = t.*

In the Ist integral and $x = \frac{\pi}{2} - y$ in the last integral, therefoe we get

$$\frac{1}{2}\int_0^{\pi} \log sint dt = \frac{\pi}{2} log2 + I + \int_{\frac{\pi}{2}}^{0} logsiny(-dy)$$

$$= \frac{\pi}{2} log2 + I + \int_0^{\frac{\pi}{2}} \log sinx dx .$$

$\therefore \quad \frac{1}{2}\int_0^{\frac{\pi}{2}} log\, sinxdx + \int_{\frac{\pi}{2}}^{\pi} log\, sinxdx \;=\; \frac{\pi}{2}log2 + 2I\,.$

This implies $\quad \frac{1}{2}[I + \int_0^{\frac{\pi}{2}} log\, sin\left(y + \frac{\pi}{2}\right) dy] = \frac{\pi}{2}log2 + 2I.$

Thus, $\quad \frac{1}{2}\left[I + \int_0^{\frac{\pi}{2}} log\, cosxdx\right] = \frac{\pi}{2}log2 + 2I$

$$= \frac{1}{2}[I + I] = \frac{\pi}{2}log2 + 2I$$

$$= \frac{\pi}{2}log2 + 2I$$

$hhhhh \quad \therefore \quad I = \frac{\pi}{2}$ *log2 + 2I*

This Implies that $I = -\frac{\pi}{2}log2\,.$

Hence $\int_0^{\frac{\pi}{2}} log\, sinxdx = \int_0^{\frac{\pi}{2}} log\, cosxdx = -\frac{\pi}{2}log2.$

Exercises.

(1) $\int_0^1 \frac{dx}{1-x}$ *dx* *(2)* $\int_0^1 \frac{x^n}{1+x}$ *dx*

(3) $\int_0^1 \frac{sinx}{x^{\frac{3}{2}}}$ *dx*

(4) $\int_1^3 \frac{x^2+1}{x^2-4}$ *dx* *(5)* $\int_0^{\pi} \frac{\sqrt{x}}{sinx}$*dx*

(6) $\int_0^1 \frac{x^n logx}{(1+x)^2}$*dx*

Answer (1) divergent (2) convergent for *n >-1* (3) convergent (4) divergent

(5) divergent (6) convergent for *n > -1*

General Test for Convergence. (Integrand May Change Sign).

We now discuss a general test for convergence of an improper integral (finite limits of integration, but discontinuous integrand) which holds whether or not integrand keeps the same sign.

Theorem 3.5 (Cauchy's Tests).

The improper integral $\int_a^b f dx$ converges at a iff to every $\varepsilon > 0$, there corresponds $\delta > 0$, such that

$$\left|\int_{a+\mu_1}^{a+\mu_2} f dx\right| < \varepsilon \qquad 0$$

$< \mu_1 < \mu_2 < \delta$.

Proof. The improper integral $\int_a^b f dx$ is said to be exists .

When , $\lim_{\mu \to 0^+} \int_{a+\mu}^b f dx$ exists finitely.

Let $F(\mu) = \int_{a+\mu}^b f dx$.

So $F(\mu)$ is a function of μ .

According to Cauchy's Criterion for finite limits $F(\mu)$ tends to a finite limit as $\mu \to 0$. If and only if to every $\varepsilon > 0$, there corresponds $\delta > 0$, such that for all possible μ_1 , $\mu_2 < \delta$;

$$|F(\mu_1) - \mu_2)| < \varepsilon$$

That is, $$\left|\int_{a+\mu_1}^b f dx - \int_{a+\mu_2}^b f dx\right| < \varepsilon$$

or $$\left|\int_{a+\mu_1}^{a+\mu_2} f dx\right| .$$

Absolute Convergence.

Definition 3.3 . The improper integral $\int_a^b f dx$ is said to be absolutely convergent if $\int_a^b |f| dx$ is convergent .

Theorem 3.6. Every absolutely convergent integral is convergent .

That is, $\int_a^b f dx$ exist if $\int_a^b |f| dx$ exist.

Proof. Since $\int_a^b |f| dx$ exist.

Therefore by Cauchy's test, to every *ε >0 ∃ δ >0*, such that

$$\left|\int_{a+\mu_1}^{a+\mu_2} |f| dx\right| < \varepsilon \ , \qquad o<\mu_1<\mu_2< \delta$$

(4)

Since $\left|\int_{a+\mu_1}^{a+\mu_2} f dx\right| \leq \int_{a+\mu_1}^{a+\mu_2} |f| dx$ (5)

and $\left|\int_{a+\mu_1}^{a+\mu_2} |f| dx\right| = \left|\int_{a+\mu_1}^{a+\mu_2} |f| dx\right|$.

Therefore (4) and (5) gives

$$\left|\int_{a+\mu_1}^{a+\mu_2} f dx\right| < \varepsilon, \qquad \forall \ \varepsilon>0, \ 0<\mu_1<\mu_2<\delta.$$

Thus , $\int_a^b f dx$ is convergent.

Alternative Method 3.6.

Since $f \leq |f|$ implies that $|f| - f \geq 0$.

Also , $|f| - f \leq 2\,|f|$ (6)

Thus, $|f| - f$ is a non- negative function on $[a, b]$ and satisfying (6).

Also $\int_a^b 2|f| dx$ is convergent. Therefore by (1) and comparison test, we get

$$\int_a^b (f - |f|) dx \text{ is convergent .}$$

This gives that $\int_a^b \{(f - |f|) + |f|\}\, dx$ is convergent .

Hence $\int_a^b f dx$ is convergent .

Example 3.9. Show that $\int_0^1 \frac{sin\frac{1}{x}}{x^p} dx$, *p>0* converges absolutely for *p<1.*

Solution. Let $f(x) = \frac{sin\frac{1}{x}}{x^p}$, $p > 0;$

'0' is the only point of infinite discontinuity and *f* does not keeps the same sign in [0, 1].

$$\therefore \qquad |f(x)| = \frac{\left|sin\frac{1}{x}\right|}{x^p} < \frac{1}{x^p}$$

Also, $\int_0^1 \frac{1}{x^p} dx$ converges for $p < 1$.

Thus $\int_0^1 \left|\frac{sin\frac{1}{x}}{x^p}\right| dx$ converges if and only if $p > 0.$

Hence $\int_0^1 \left|\frac{sin\frac{1}{x}}{x^p}\right| dx$ is absolutely convergent if and only if $p < 1$.

Infinite range of integration.

We shall now consider the convergence of improper integral of bounded integrable function with infinite range of integration (a or b both infinite).

Definition 3.4. (Convergence at ∞).

The symbol $\int_a^\infty f dx,$ $\qquad x \geq a$ (7)

is defined as limit of $\int_a^X f dx$ when $X \to \infty$, so that

$$\int_a^\infty f dx = \lim_{X\to\infty} \int_a^X f dx \qquad (8)$$

If the limit exists and is finite then the improper integral (8) is said to be divergent.

Note. For $a_1 > a$, $\int_a^X f dx = \int_a^{a_1} f dx + \int_{a_1}^X f dx$

which implies that the integrals $\int_a^\infty f dx \; and \; \int_{a_1}^\infty f dx$ are either both convergent or both divergent.

Exercises.

(i) $\int_0^\infty \frac{xdx}{1+x^2}$ *(ii)* $\int_1^\infty \frac{dx}{\sqrt{x}}$

(iii) $\int_a^\infty sinxdx$.

Solution. (i) For *X > 0, we have*

$$\int_0^X \frac{xdx}{1+x^2} = \frac{1}{2}\int_0^X \frac{2xdx}{1+x^2}$$

$$= \frac{1}{2}[log(1+x^2)]_0^X$$

$$= \frac{1}{2}[log(1+x^2)]$$

Clearly , $\lim_{X\to\infty} \int_0^X \frac{xdx}{1+x^2} = \infty$

Hence $\int_0^\infty \frac{xdx}{1+x^2}$ is divergent.

Solution (iii) We have

$$\int_a^X sinxdx = (-cosx)_a^X \quad X > a$$

$$= cos\, a - cosX$$

Clearly, $\lim_{X\to\infty}(cosa - cosX)$ exists finitely but not uniquely.

Thus, $\lim_{X\to\infty} \int_a^X sinxdx$ does not exit .

Hence $\int_a^\infty sinxdx$ diverges.

Convergence at $-\infty$.

$$\int_{-\infty}^{b} f dx, \qquad x \leq b \tag{9}$$

is defined by equation

$$\int_{-\infty}^{b} f dx, \quad = \lim_{X \to -\infty} \int_{X}^{b} f dx \tag{10}$$

If the limit exists and is finite the integral (9) converges otherwise it diverges.

Convergence at both ends.

$$\int_{-\infty}^{\infty} f dx \quad , \forall \; x \tag{11}$$

Is understood to mean

$$\int_{-\infty}^{c} f dx + \int_{c}^{\infty} f dx \tag{12}$$

where c is any real number .

If both integrals in (12) converges according to definition (I) and (II), then, the integral $\int_{-\infty}^{\infty} f dx$ also converges, otherwise it diverges.

Exercises.

Examine for convergence the integrals

$(I) \int_{0}^{\infty} sinx dx$ $\qquad (II) \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ $\qquad (III) \int_{2}^{\infty} \frac{2x^2 dx}{x^4-1}$

$(IV) \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$ $\qquad (V) \int_{0}^{\infty} x^3 e^{-x^2} dx$

Solution :-(I) try yourself (limit does not exist)

Solution:-(II) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{\substack{X \to \infty \\ Y \to -\infty}} \int_{Y}^{X} \frac{dx}{1+x^2}$

$$= \lim_{\substack{X \to \infty \\ Y \to -\infty}} (tan^{-1}x)_{Y}^{X}$$

$$= \lim_{\substack{X\to\infty \\ Y\to-\infty}} (tan^{-1}X - tan^{-1}Y)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi \ .$$

Thus the integral converges and is equal to π .

(III) $\int_2^\infty \frac{2x^2 dx}{x^4-1} = \lim_{X\to\infty} \int_2^X 2x^2\, dx$

$$= \lim_{X\to\infty} [tan^{-1}x - tan^{-1}2 + \frac{1}{2} log \frac{X-1}{X+1} + \frac{1}{2} log3]$$

$$= \frac{\pi}{2} - tan^{-1}2 + \frac{1}{2} log3 \ .$$

Thus the integral converges.

(IV) $\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = 2\int_0^\infty \frac{dx}{(x^2+1)^2}$

$$= 2 \lim_{X\to\infty} 2 \int_0^X \frac{dx}{(x^2+1)^2}$$

$$= \lim_{X\to\infty} [tan^{-1}x + \frac{x}{1+x^2}] \ .$$

By Putting $x = \tan\theta$

= π/2

(V) $\int_0^\infty x^3 e^{-x^2}\, dx$ = ½ , converges.

Comparison test for convergence at ∞.

Theorem 3.7. A necessary and sufficient condition for the convergence of $\int_a^\infty f dx$, where f is positive in $[a, \infty)$, that there exists a positive number M, independent of X, such that

$$\int_a^X f dx < M, \ \forall X \geq a.$$

Proof. The integral $\int_a^X f dx$ is said to be convergent if $\int_a^X f dx$ tends to a finite limit as $X \to \infty$. Since f is positive in [a, X] , $\forall X \geq a$ and $\int_a^X f dx$ is monotonic increasing function on X i.e. $\int_a^X f dx$ increases as X increases.

Also since $\int_a^X f dx < M, \; for \; some \; m > 0 \; and \; \forall \; X \geq a$.

That is, $\int_a^X f dx$ is bounded above.

Therefore, $\lim_{X \to 0} \int_a^X f dx$ exist finitely.

Conversely, suppose $\int_a^\infty f dx$ is convergent, then $\lim_{X \to \infty} \int_a^X f dx \; exists \; finitely$.

Therefore, $\exists M > O$, such that $\forall X \geq a$

$$\int_A^X f dx < M$$

as $\int_A^X f dx$ increases as X increases.

Hence the theorem is proved completely.

Comparison Test I.

Theorem 3.8. If f and g are positive and $f(x) \leq g(x)$, for all $x \in [a, b]$.

Then, (I) $\int_a^\infty f dx$ converges if $\int_a^\infty g dx$ converges.

(II) $\int_a^\infty g dx \; diverges$ if $\int_a^\infty f dx$ diverges.

Proof. Suppose $\int_a^\infty g dx$ converges.

Therefore $\exists M > 0$ such that $\forall X \geq a$,

$$\int_a^X g dx < M.$$

This gives $\int_a^X f dx < M$

Hence $\int_a^\infty f dx$ converges .

(II) Suppose $\int_a^\infty f dx$ diverges then $\exists\, X_1$, such that

$$\int_a^{X_1} f dx > M, \ \forall M>0$$

This implies that $\int_a^{X_1} g dx > M, \ \forall M>0$

This gives $\int_a^\infty g dx$ diverges.

Note. Since *f* and g are bounded in $[a, X]$.

Therefore, $f(x) \le g(x)$.

This implies that $\int_a^X f dx \le \int_a^X g dx \quad \forall \ X \ge a$.

Comparison Test –II.

Theorem 3.9. If f and g are positive functions in $[a$, X] and

$$\lim_{X\to\infty} \frac{f(x)}{g(x)} = l \quad (\neq 0),$$

then two integrals converges or diverges together.

Also if $\lim_{X\to\infty} \frac{f}{g} = 0$ *and* $\int_a^\infty g\, dx$ converges, then $\int_a^\infty f dx$ converges and if $\lim_{X\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty g\, dx$ diverges, then $\int_a^\infty f dx$ also diverges.

Proof. Evidently $l > 0$ choose > 0 , such that $l - \varepsilon > 0$

Since $\lim_{X\to\infty} \frac{f(x)}{g(x)} = l$

Therefore $\forall\, \varepsilon > 0, \exists\ k > 0$ such that

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon \quad \text{whenever } |x| > k .$$

That is $\quad l-\varepsilon < \frac{f(x)}{g(x)} < l+\varepsilon \qquad \forall\, \varepsilon > 0, with\ x > k$

$$(l-\varepsilon)g(x) < f(x) \tag{13}$$

$$f(x) < (l+\varepsilon)g(x) \tag{14}$$

for $x > k\ and\ \forall\, \varepsilon > 0.$

Clearly $\ l-\varepsilon > 0$, by choosing ε so small.

Therefore by comparison test and (13) and (14) we get

$\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f dx$ converges.

Again

$$\lim_{X\to\infty} \frac{f(x)}{g(x)} = 0$$

implies that $\quad f(x) < g(x), \qquad \forall\, x > k$

Therefore if $\int_a^\infty f dx$ is divergent, then $\int_a^\infty g dx$ is convergent and if $\int_a^\infty g dx$ is convergent then $\int_a^\infty f dx$ is convergent.

Also if, $\quad \lim_{X\to\infty} \frac{f(x)}{g(x)} = \infty$

This implies $\quad \frac{f(x)}{g(x)} > M, \ \forall\ x > k$

Therefore $\quad f(x) > Mg(x), \quad \forall\ x > k$

Hence if $\int_a^\infty g dx$ is divergent, then $\int_a^\infty f dx$ is divergent.

Useful Comparison Integral.

Theorem 3.10. Show that the improper integral $\int_a^\infty f dx = \int_a^\infty \frac{c}{x^n} dx$, *a>0* where c is a positive constant, converges if and only if $n > 1$.

Proof. We have

$$\int_a^\infty \frac{c}{x^n}dx = \begin{Bmatrix} c\,log\frac{x}{a} & , \quad n=1 \\ \frac{1}{1-n}\left[\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right], & n \neq 1 \end{Bmatrix}$$

$$\therefore \quad \lim_{X\to\infty} \int_a^X \frac{c}{x^n}dx = \begin{Bmatrix} \infty & , & if\ n \leq 1 \\ \frac{c}{(n-1)^{n-1}} & & if \quad n > 1 \end{Bmatrix}.$$

Thus, $\int_a^\infty \frac{c}{x^n}dx$ converges if and only if n>1.

From this useful integral and comparison test, the improper integral $\int_a^\infty f dx$ $c$onverges if there exists a positive number $n > 1$ such that

$f(x) \leq \frac{M}{x^n}$ for some $M > 0$ and for some all $x \geq a$.

Also if, $\lim_{x\to\infty} x^n f(x)$ exists and is non- zero, then integral $\int_a^\infty f dx$ converges if and only if $n > 1$.

Exercises.

(I) $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

(II) $\int_0^\infty \frac{x^2dx}{\sqrt{x^5+1}}$

(III) $\int_0^\infty e^{-x^2}dx$

(IV) $\int_0^\infty \frac{logx}{x^2}dx$

(V) $\int_1^\infty x^n e^{-x}dx$

(VI) $\int_0^\infty \frac{sin^2x}{x^2}dx$

Solution :- (I) Take $f(x) = \frac{dx}{x\sqrt{x^2+1}}$ and

$$g(x) = \frac{1}{x^2}$$

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1(\neq 0)$

Thus $\int_1^\infty f dx = \int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges.

(II) Let $f(x) = \frac{x^2dx}{\sqrt{x^5+1}}$

Take $g(x = \frac{1}{\sqrt{x}}$,

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 \ (\neq 0)$

Thus $\int_0^\infty \frac{x^2 dx}{\sqrt{x^5}+1}$ diverges.

(IV) Let $f(x) = \frac{logx}{x^2}$

Take $g(x) = x^{\frac{3}{2}}$,

Then $\lim_{X\to\infty} \frac{x^{\frac{3}{2}}log}{x^2} = \lim_{x\to\infty} \frac{logx}{x^{\frac{1}{2}}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$\lim_{x\to\infty} 2\left(\frac{1}{x^{\frac{1}{2}}}\right) = 0$$

Since $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}$ is convergent.

Therefore $\int_0^\infty \frac{logx}{x^2}dx$ is convergent.

(V) Let f(x) = $x^n e^{-x}$dx

Take $g(x) = x^2$.

Then, $\lim_{x\to\infty} x^2 . x^n e^{-x} = \lim_{x\to\infty} (n+2)!\, e^{-x} = 0$ and

$\int_1^\infty \frac{1}{x^2} dx$ is convergent.

Therefore $\int_1^\infty x^n e^{-x} dx$ is convergent.

(III) Let $f(x) = \int_0^\infty e^{-x^2} dx$.

Clearly 0 is not point of infinite discontinuity, we may write

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx \quad + \quad \int_1^\infty e^{-x^2}dx = I_1 + I_2 \ .$$

Clearly I_1 is proper and I_2 is improper integral.

We test for $I_2 = \int_1^\infty e^{-x^2}dx$.

We have $e^{-x^2} > x^2 \quad \forall\, x \in R$

$\frac{1}{e^{-x^2}} < \frac{1}{x^2} \quad \forall\, x \in R$

This implies that $e^{-x^2} < \frac{1}{x^2} \quad \forall\, x \in R$.

Again , $\int_1^\infty \frac{1}{x^2}dx$ is convergent.

Therefore, $\int_1^\infty e^{-x^2}dx$ is convergent.

Hence $\int_0^\infty e^{-x^2}dx$ is convergent.

(VI) $\int_0^\infty \frac{sin^2x}{x^2}dx$ is convergent because $sin^2x \le 1, \quad \forall\, x \in R$.

Exercises.

(I) $\int_0^\infty \frac{xtan^{-1}x}{(1+x^4)^{\frac{1}{3}}}dx$ $\quad$ (II) $\int_{e^2}^\infty \frac{dx}{xloglogx}$ $\quad$ *(III)* $\int_0^\infty (\frac{1}{x} - \frac{1}{sinhx})$.

Solution (I). Let $f(x) = \frac{xtan^{-1}x}{(1+x^4)^{1/3}}$ $\quad$ *($\sim x^{-\frac{1}{3}}$)*

Take $g(x) = \frac{1}{x^{\frac{1}{3}}}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \pi/2$.

Since $\int_0^\infty \frac{1}{x^{\frac{1}{3}}}dx$ *is divergent.*

Therefore $\int_0^\infty f dx$ is divergent.

(II) Put $logx = t$, we get

$$\int_{e^2}^\infty \frac{dx}{xloglogx} = \int_2^t \frac{dt}{logt}$$

$$= \int_2^\infty \frac{dx}{logx}$$

Let $f(x) = \frac{1}{logx}$

Take $g(x) = \frac{1}{x^m}$, then

$$\lim_{x\to\infty} \frac{x^m}{logx} = \lim_{x\to\infty} \frac{x}{logx} \text{ by taking } m = 1$$

Therefore $\lim_{x\to\infty} \frac{x}{logx} = \lim_{x\to\infty} x = \infty$.

Since $\int_2^\infty \frac{dx}{x}$ is divergent, so that $\int_2^\infty \frac{dx}{logx}$ is also divergent .

Hence $\int_{e^2}^\infty \frac{dx}{xloglogx}$ is divergent.

(I) $f(x) = (\frac{1}{x} - \frac{1}{sinhx})/x$

Clearly 0 is not point of infinite discontinuity, because

$$\lim_{x\to 0^+} f(x) = \frac{1}{6}$$ (By L Hospital's rule)

We have $f(x) = \left(\frac{1}{x} - \frac{1}{sin^h x}\right)\frac{1}{x}$

$$= \frac{1}{x^2} - \frac{1}{xsinhx}$$

$$= \frac{1}{x^2} - \frac{1}{x}\left[\frac{1}{e^x - e^{-x}/\ 2}\right]$$

$$= \frac{1}{x^2} - \frac{1}{x}\left[\frac{2}{e^x - e^{-x}}\right]$$

$$= \frac{1}{x^2} - \frac{1}{x}\left[\frac{2e^{-x}}{1 - e^{-2x}}\right].$$

Take $g(x) = \frac{1}{x^2}$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l \quad (\neq 0)$$

Thus, $\int_0^\infty f dx$ is convergent.

Note. $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} x^2\left[\frac{1}{x^2} - \frac{1}{x}\frac{2e^{-x}}{1-e^{-2x}}\right]$

$$= \lim_{x\to\infty}\left[1 - \frac{2xe^{-x}}{1-e^{-2x}}\right].$$

We have $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$

$$= \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Therefore $\lim_{x\to\infty}\left[1 - \frac{2xe^{-x}}{1-e^{-2x}}\right] = 1 - \frac{0}{1-0} = 1\ (\neq 0).$

Example 3.10. (Gamma Function).

The integral $\int_0^\infty x^{m-1}\, e^{-x} dx$ is convergent if and only if $m > 0$.

Solution. Let $f(x) = x^{m-1}e^{-x}$.

If $m < 1$, the '0' infinite discontinuity.

So we must examine the convergence of above improper integral at both 0 and ∞.

$$\int_0^\infty x^{m-1}\, e^{-x} dx = \int_0^1 x^{m-1}\, e^{-x} dx + \int_1^\infty x^{m-1}\, e^{-x} dx$$

Convergence at 0 for $m < 1$:

Let $g(x) = \frac{1}{x^{1-m}}$,

Then, $\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-x} = 1\ (\neq 0).$

Since $\int_0^1 \frac{1}{x^{1-m}} dx$ converges, if and only if m>0.

Therefore $\int_0^1 x^{m-1}\, e^{-x} dx$ converges if and only if $m > 0$.

Converges at ∞.

Let $g(x) = \frac{1}{x^2}$, so that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{m+1} / e^{x}$$

$$= \lim_{x\to\infty} \frac{(m+1)!}{e^{x}} = 0.$$

Since $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

Thus, $\int_1^\infty x^{m-1} e^{-x} dx$ is convergent $\forall\, m$.

Hence $\int_1^\infty x^{m-1} e^{-x} dx$ is convergent if and only if $m > 0$ and is denoted by $< m)$.

Thus, $\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx$, $m>0$.

Thus Γ*(0), Γ(-1),* etc. are not exists .

Example 3.11. Examine for the convergence of $\int_3^\infty \frac{dx}{x^2+x-2}$ and

$G(x) = \frac{1}{x^2}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x^2+x-2}$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}-\frac{2}{x^2}}$$

$$= 1 .$$

Thus, $\int_3^\infty \frac{dx}{x^2+x-2}$ is convergent.

Again let us decompose the integrand into partial fraction.

We have $\frac{1}{x^2+x-2} = \frac{1}{3(x-1)} - \frac{1}{3(x+2)}$.

It is obvious $\int_3^\infty \frac{1}{3(x-1)}\, dx$ *and* $\int_3^\infty \frac{1}{3(x+2)} dx$ are both divergent .

Thus, $\int_3^\infty \frac{dx}{x^2+x-2} = \int_3^\infty \frac{1}{3(x-1)}\, dx + \int_3^\infty \frac{-dx}{3(x+2)}$ is not correct.

Now we evaluate above improper integral.

We have $\int_3^\infty \frac{dx}{x^2+x-2} = \lim_{x\to\infty} \int_3^x \frac{dx}{x^2+x-2}$

$$=\lim_{x\to\infty}\left[\int_3^x \frac{dx}{3(x-1)}-\int_3^x \frac{dx}{3(x+2)}\right]$$

$$=\lim_{x\to\infty}\left[\frac{1}{3}\{log(x-1)-log(x+2\}\,{}_3^x\right]$$

$$=\lim_{x\to\infty}\frac{1}{3}\left[log(\frac{x-1}{x+2})\right]{}_3^x$$

$$=\lim_{x\to\infty}\frac{1}{3}\left[log\left(\frac{x-1}{x+2}\right)\right]-lo\,g\left(\frac{2}{5}\right)$$

$$=\lim_{x\to\infty}\frac{1}{3}\left[log\left[\frac{5(x-1)}{2(x+2)}\right]\right]$$

$$=\lim_{x\to\infty}\frac{1}{3}\left[\frac{log[5-\frac{5}{x}]}{2+\frac{4}{x}}\right]$$

$$=1/3\,log\,5/2\quad.$$

General test for convergence at ∞ (Integrand may change sign).

Theorem 3.11. (Cauchy's Test).

The integral $\int_a^X f dx$ converges at ∞ if and only if to every > 0 , $\exists\ X_0$, such that

$$\left|\int_{X_1}^{X_2} f dx\right| < \epsilon \quad \forall \ X_1, X_2 > X_0 \ .$$

Proof. The improper integrand $\int_a^\infty \mathrm{fdx}$ exists if $\lim_{x\to\infty}\int_a^x \mathrm{fdx}$ exists finitely

Let $F(X) = \int_a^\infty f dx$, a function of X .

According to Cauchy's criterion for finite limits, $F(x)$ tends to a finite limits as $x \to \infty$ if to a finite $\epsilon > 0\ \exists\ X_0$, such that $\forall\ X_1, X_2 > X_0$

$$|F(X_1) - F(X_2)| < \epsilon$$

That is $\left|\int_{X_1}^{X_2} f dx\right| < \epsilon\,.$

Example 3.12. Show that $\int_0^\infty \frac{sinx}{x} dx$ is convergent .

Solution. Since $\lim_{x \to 0} \frac{sinx}{x}$ = *1.*

Therefore '0' is not infinite discontinuity, we may put

$$\int_0^\infty \frac{sinx}{x}\, dx \quad = \quad \int_0^1 \frac{sinx}{x}\, dx + \int_1^\infty \frac{sinx}{x}\, dx \ .$$

We now test for the convergence of $\int_1^\infty \frac{sin}{x}\, dx$ as $\int_0^1 \frac{sinx}{x}\, dx$ is proper integral. For any $\epsilon > 0$,

Let x_1, x_2 be two numbers both greater than $\frac{2}{\epsilon}$,

Now $\int_{x_1}^{x_2} \frac{sinx}{x} dx \quad = \left[-\frac{cosx}{x}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{cosx}{x^2} dx$

so that, $\left|\int_{x_1}^{x_2} \frac{sinx}{x} dx\right| \quad \leq \quad \left|\frac{cosx_1}{x_1} - \frac{cosx_2}{x_2}\right| + \left|\int_{x_1}^{x_2} \frac{cosx}{x^2} dx\right|$

$$\leq \ \frac{1}{x_1} + \frac{1}{x_2} + \int_{x_1}^{x_2} \frac{dx}{x^2}$$

$$= 2.\frac{\epsilon}{2} \ = \epsilon \ .$$

Therefore, by Cauchy's test the improper integral $\int_0^\infty \frac{sinx}{x} dx$ is convergent.

Absolute Convergence.

Definition 3.4. The improper integral $\int_a^\infty f dx$ is said to be absolutely convergent if $\int_a^\infty |f|\, dx$ is convergent.

Theorem 3.12.

Absolute convergences of $\int_a^\infty f dx$ implies convergence of $\int_a^\infty f dx$

i.e., $\int_a^\infty f dx$ exists if $\int_a^\infty |f|\, dx$ exist.

Proof. Suppose $\int_a^\infty |f|\, dx$ exists, then by Cauchy's Test, $\forall\, \varepsilon > 0$, $\exists$ $X_0\,, such\ that$

$$\left|\int_{x1}^{x2} |f| dx\right| \; < \; \epsilon \qquad , \; x_1, x_2, > x_0.$$

We have $\left|\int_{x_1}^{x_2} f dx\right| \quad \leq \quad \int_{x_1}^{x_2} |f| dx \quad < \quad \epsilon \qquad x_1, x_2, > x_0$.

Thus by Cauchy's test $\int_a^\infty f dx$ converges.

Example 3.13. Show that $\int_1^\infty \frac{sinx}{x^p}\, dx$ converges absolutely if $p\ > 1$

Solution. We have $\left|\frac{sinx}{x^p}\right| \; = \; \frac{|sinx|}{x^p} \; \leq \; \frac{1}{x^p}$, $\forall\ x\ \geq 1$,

and $\int_1^\infty \frac{1}{x^p} dx$ converges for $p\ >\ 1$.

Thus, $\int_1^\infty \left|\frac{sinx}{x^p}\right|\, dx$ converges for $p > 1$.

Therefore, $\int_1^\infty \frac{sinx}{x^p}\, dx$ converges absolutely for $p > 1$.

Integrand as a product of functions (convergent at '∞').

A test for absolutely convergence.

Theorem 3.13. If a function φ is bounded in $[a, \infty]$ and integrable in $[a, x]$, $\forall\ x\ \geq\ a$.

Also if $\int_a^\infty f dx$ is absolutely convergent at ∞, then $\int_a^\infty f\varphi dx$ is also absolutely convergent at ∞.

Proof. Since f is bounded in $[a, \infty)$, therefore $\exists\ k > 0$, such that

$$|\varphi(x)| \quad \leq \quad k, \;\; \forall\ x \in [a, \infty) \qquad (15)$$

Again since |f|is positive in $[a, \infty)$,and $\int_a^\infty |f| dx$ is convergent.

Therefore we can find m, such that

$$\int_a^x |f| dx \; < m, \quad \forall \;\; x \geq a \qquad (16)$$

Using (1) we have

$$|f\varphi| = |f||\varphi|$$

$$\leq \; k|f|, \qquad \forall \; x \in [a, \infty).$$

Therefore, $\int_a^x |f\varphi| dx \; \leq k \int_a^x |f| dx$

$$< \; km \qquad \forall \;\; x \geq a\,.$$

Thus, $\int_a^x |f\varphi| dx \; \leq \; km \quad \forall \;\; x \geq a\,.$

Therefore, $\int_a^x |f\varphi| dx$ is convergent.

Hence $\int_a^x f\varphi dx$ is absolutely convergent.

Test for convergence.

Theorem (Abel's Test) 3.14. If φ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f dx$ is convergent at ∞, then, $\int_a^\infty f\varphi dx$ is convergent at ∞.

Proof. Since φ is monotonic in $[a, \infty), then$ is integrable in $[a, x]$, $\forall \;\; X \geq a\,.$

Also since f is integrable in $[a, x]$, we have by 2nd mean value theorem

$$\int_{X_1}^{X_2} f\varphi \, dx = \varphi(X_1) \int_{X_1}^{y} f dx + \varphi(X_2) \int_{y}^{X_2} f dx \qquad (17)$$

$$for \quad a < X_1 < Y \leq X_2.$$

Let $\epsilon > 0$ be arbitrary.

Since φ is bounded in $[a, \infty)$, a positive number k exists, such that

$$|\varphi(x)| \le k, \quad \forall \; X \ge a \; .$$

In particular,

$$|\varphi(x_1)| \le k, \qquad |\varphi(x_2)| \le k, \qquad (18)$$

Again since $\int_a^\infty f dx$ is convergent, therefore their exists X_0, such that

$$\left|\int_{x_1}^{x_2} f dx\right| < \frac{\epsilon}{2k} \quad , \quad \forall \; X_1, X_2 > X_0 \qquad (19)$$

Since, $X_1 \le Y \le X_2$.

Therefore, $\left|\int_{x1}^{y} f dx\right| < \frac{\epsilon}{2k}$ and

$$\left|\int_{y}^{X_2} f dx\right| < \frac{\epsilon}{2k} \qquad (20)$$

Thus from (17), (18),(19), and (20), we deduce that $\exists$ X_0, such that for all , $X_1, X_2 > X_0$ and $\epsilon > 0$

$$\left|\int_{X_1}^{X_2} f\varphi dx\right| \le |\varphi(x1)| \left|\int_{X_1}^{y} f dx\right| + |\varphi(x2)| \left|\int_{y}^{X_2} f dx\right| < k\frac{\epsilon}{2k} + k\frac{\epsilon}{2k} = \epsilon \; .$$

Hence $\int_a^\infty f\varphi dx$ is convergent.

Theorem Drichlet's Test 2.15. If φ is bounded and monotonic in $[a, \infty)$ and tends to 0 as $x \to \infty$ and $\int_a^x f dx$ is bound for $X \ge a$, then $\int_a^\infty f\varphi dx$ convergent at ∞.

Proof. Since φ is bounded and integrable in $[a, x]$. Also since f is integrable in $[a, x]$, therefore by second mean value theorem:

$$\int_{x_1}^{x_2} f\varphi dx = \varphi(x_1)\int_{x_1}^{y} f dx + \varphi(x_2)\int_{y}^{x_2} f dx \qquad (21)$$

for $a < X_1 \le Y \le X_2$.

Again, since $\int_a^x f dx$ is bound when $X \ge a$, therefore $\exists\, k$, such that

$$\left|\int_a^x f dx\right| \le k, \quad \forall \; X \ge a \; .$$

Therefore,

$$\left| \int_{x_1}^{y} f dx \right| = \left| \int_a^y f dx - \int_a^{x_1} f dx \right|$$

$$\leq \left| \int_a^y f dx \right| + \left| \int_a^{x_1} f dx \right|$$

$$\leq 2k, \quad \text{for } X_1, \geq a .$$

Similarly,

$$\left| \int_y^{x_2} f dx \right| \leq 2k, \quad \text{for } X_2 \geq a .$$

Let $\epsilon > 0$ be arbitrary.

Since $\varphi \to 0$ as $\to \infty$, there exists a positive X_0, such that

$$|\varphi(X_1)| < \frac{\epsilon}{4k}, \quad |\varphi(X_2)| < \frac{\epsilon}{4k} \quad \text{where} \quad X_2 \geq X_1 \geq X_0 .$$

Let the numbers X_1, X_2 in (21) be $\geq X_0$, so that from (17), (18), (19) & (20), we get

$$\left| \int_{X_1}^{X_2} f\varphi dx \right| < \frac{\epsilon}{4k} 2k + \frac{\epsilon}{4k} 2k$$

$$= \epsilon \quad \forall \; X_2 \geq X_1 \geq X_0 .$$

Hence by Cauchy's test $\int_a^\infty f\varphi dx$ is convergent at ∞.

Example 3.14. The improper integral $\int_1^\infty \frac{sinx}{x^p} dx$ is divergent for p>0.

Solution. Take φ*(x)* = $\frac{1}{x^p}$, $p > 0$ and

$$f(x) = sinx .$$

Then $\varphi(x)$ is monotonic decreasing and tends to 0 as $x \to \infty$.

Also, $\left| \int_1^x f dx \right| = \left| \int_1^x sinx \, dx \right|$

$$= |cos1 - cosx|$$

$\leq |cos1|+|cosx|$

$\leq 1+1 = 2, \quad \forall X \geq 1$.

Thus, $\left|\int_1^x sinxdx\right| \leq 2 \quad \forall X \geq 2$.

Therefore $\int_1^x sinx\, dx$ is bounded.

Hence by Drichlet's test $\int_1^\infty sinx \frac{1}{x^p} = \int_1^\infty \frac{sinx}{x^p} dx$ is convergent $p > 0$.

Also, we know that $\int_1^\infty \frac{sinx}{x^p} dx$ is absolutely convergent if and only if $p > 1$

Thus, $\int_1^\infty \frac{\mathrm{sinx}}{\mathrm{x^p}} \mathrm{dx}$ is conditional convergent for $0 < p \leq 1$.

Conditionally Convergent.

An improper integral $\int_a^\infty f dx$ is conditionally convergent at ∞ if $\int_a^\infty f dx$ is convergent at ∞, but $\int_a^\infty |f| dx$ is not convergent .That is the improper integral is said to be conditionally convergent if it is convergent but not absolutely.

Example 3.15. Show that $\int_0^\infty \frac{sinx}{x} dx$ is convergent, but not absolutely.

Solution. We have $\int_0^\infty \frac{sinx}{x} dx = \int_0^1 \frac{sinx}{x} dx + \int_1^\infty \frac{sinx}{x} dx$

Now , $\int_0^1 \frac{sinx}{x} dx$ is proper integral.

To examine the convergence of $\int_1^\infty \frac{sinx}{x} dx$ at ∞, we see that

$$\left|\int_1^X sinxdx\right| = |cos1 - cosX| \leq |cos1| + |cosX| < 2$$, so that

$\left|\int_1^X sinxdx\right|$ is bounded above for all $X \geq 1$.

Also, $1/x$ is a monotonic decreasing function tending to 0 $as\ x \to \infty$.

Therefore by Dirchlet's test $\int_0^\infty \frac{sinx}{x}\,dx$ is convergent.

Hence $\int_0^\infty \frac{sinx}{x}\,dx$ is convergent.

To show that $\int_0^\infty \frac{sinx}{x}\,dx$ is not absolutely convergent, we proceed as follows:

$$\int_0^{n\pi} \left|\frac{sinx}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|sinx|}{x} dx$$

Now, $\forall\, x \in [(r-1)\pi,\ r\pi]$

$$\int_{(r-1)\pi}^{r\pi} \frac{|sinx|}{x} dx \geq \int_{(r-1)\pi}^{r\pi} \frac{|sinx|}{r\pi} dx$$

Putting, $x = (r-1)\pi + y$

$$\int_{(r-1)\pi}^{r\pi} \frac{|sinx|}{r\pi} dx = \int_0^{\pi} \frac{|sin(r-1)\pi + y|dy}{r\pi}$$

$$= \frac{1}{\pi r} \int_0^{\pi} sinydy = \frac{2}{r\pi}.$$

Hence $\int_0^{n\pi} \frac{|sinx|}{x} dx = \sum_{i=r}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|sinx|}{x} dx \geq \sum_{i=1}^{n} \frac{2}{r\pi}$.

But $\sum_{r=1}^{n} \frac{2}{r\pi}$ is a divergent series.

Therefore, $\lim_{n\to} \int_0^{n\pi} \frac{|sinx|}{x} dx \geq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{r\pi}$.

This implies that $\lim_{n\to\infty} \int_0^{n\pi} \frac{|sinx|}{x} dx$ is infinite.

Now, let t be a real number, there exists positive integer n , such that

$$n\pi \leq t < (n+1)\pi .$$

We have, $\int_0^{t} \frac{|sinx|}{x} dx \geq \int_0^{n\pi} \frac{|sinx|}{x} dx.$

Let $t \to \infty$, so that $n \to \infty$, thus we see that

$$\int_0^{t} \frac{|sinx|}{x} dx \to \infty .$$

This implies $\int_0^\infty \frac{|sinx|}{x} dx$ does not converge.

This example show that $\int_o^\infty \frac{sinx}{x^p} dx$, $0 < p \leq 1$, is convergent but not absolutely .

Example 3.16. Show that $\int_2^\infty \frac{cosx}{logx} dx$ is conditionally convergent.

Solution. Let $\varphi(x) = \frac{1}{logx}$, $f(x) = cosx$.

$$\left|\int_2^X cosxdx\right| = |sinX - sin2| \leq |sinX| + |sin2| \leq 2,$$ so that

$\int_2^X cosx\, dx$ is bounded for all $X \geq 2$

Also, $\varphi(x) = \frac{1}{logx}$ is monotonic decreasing function tending to 0 as $x \to \infty$.

Hence by Dirichlet's test $\int_2^\infty \frac{cosx}{logx} dx$ is convergent.

For absolute convergence consider

$$I= \int_2^\infty \left|\frac{cosx}{logx}\right| dx = \int_2^{\frac{3\pi}{2}} \frac{|cosx|}{logx} dx + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{|cosx|}{logx} dx-----$$

$$+\int_{\frac{(2n-1)\pi}{2}}^{\frac{(2n+1)\pi}{2}} \frac{|cosx|}{logx} dx+----$$

Therefore,

$$I= \int_{\frac{\pi}{2}}^{2} \frac{|cosx|}{logx} dx+ \int_2^{\frac{3\pi}{2}} \frac{|cosx|}{logx} dx+-----\int_{\frac{(2n-1)\pi}{2}}^{\frac{(2n+1)\pi}{2}} \frac{|cosx|}{logx} dx+ .....$$

$$-\int_{\frac{\pi}{2}}^{2} \frac{|cosx|}{logx} dx$$

$$= \sum_{r=1}^{n} \int_{\frac{(2n-1)\pi}{2}}^{\frac{(2n+1)\pi}{2}} \frac{|cosx|}{logx} dx - \int_{\frac{\pi}{2}}^{2} \frac{|cosx|}{logx} dx$$

Now,

$$\int_{\frac{(2n-1)\pi}{2}}^{\frac{(2n+1)\pi}{2}} \frac{|cosx|}{logx} dx \geq \frac{1}{log(2r+1)\pi/2} \left|\int_{\frac{(2r-1)\pi}{2}}^{\frac{(2r+1)\pi}{2}} cosxdx\right|$$

$$= \frac{1}{log\left[\frac{(2r+1)\pi}{2}\right]} \left|sin\left[(2r+1)\frac{\pi}{2}\right] - sin\left[(2r-1)\frac{\pi}{2}\right]\right|$$

$$= \frac{|2(-1)r|}{log\left[\frac{(2r+1)\pi}{2}\right]}$$

$$= \frac{2}{log\frac{(2r+1)\pi}{2}} .$$

Therefore, $I \geq \sum_{r=1}^{\infty} \frac{2}{log\frac{(2r+1)\pi}{2}} - \int_{\frac{\pi}{2}}^{2} \frac{|cosx|}{logx} dx$.

But $\sum_{r=2}^{\infty} \frac{1}{logx}$ is divergent and $\int_{\frac{\pi}{2}}^{2} \frac{|cosx|}{logx} dx$ is proper integral.

Hence $I=\int_0^{\infty} \frac{|cosx|}{logx} dx$ is divergent and so $\int_2^{\infty} \frac{cosx}{logx} dx$ is conditionally convergent.

Example 3.17. Using $\int_0^{\infty} \frac{sinx}{x} dx = \frac{n}{2}$, show that

$$\int_0^{\infty} \frac{sin^2x}{x^2} dx = \frac{\pi}{2} .$$

Solution. To compute it let us integrate by parts, therefore

$$\int_0^{\infty} \frac{sin^2x}{x^2} dx = \left[\frac{-sin^2x}{X}\right]_0^{\infty} + \int_0^{\infty} \frac{sin2x}{X} dx$$

Hence $$\int_0^{\infty} \frac{sin^2x}{x^2} dx = \int_0^{\infty} \frac{sint}{t} dt = \frac{\pi}{2} .$$

Example 3.18. The function f is defined on [0, ∞ [by f(x) = $(-1)^{n-1}$, $for \ \ n-1 \leq x < n, \ n \in N$, show that the integral $\int_0^{\infty} f(x)dx$ does not converge.

Solution. Consider

$$\int_0^{2n} f(x)dx = \int_0^1 (-1)^0 dx + \int_1^2 (-1)dx +$$

$$+ \int_2^3 (-1)^2 dx + \cdots \int_{2n-1}^{2n} (-1)^{2n-1} dx$$

$$=1-1+1-1+1-1 \ldots\ldots\ldots..+1-1.$$

and

$$\int_0^{2n+1} f(x)dx \;=\; \int_0^1 dx + \int_1^2 (-1)dx \;+\ldots \quad \ldots\ldots \int_{2n}^{2n+1} (-1)^{2n}\,dx$$

$$=1-1+1 \ldots\ldots\ldots..-1+1 \;\; =1$$

$$\therefore \quad \lim_{n\to\infty} \int_0^{2n} f(x)dx \;=\; 0$$

and $\lim_{n\to\infty} \int_0^{2n+1} f(x)dx \;=\; 1.$

Hence the integral does not exist and therefore it is not convergent.

Example 3.19. Test the convergence of

(I) $\int_0^\infty \frac{xdx}{1+x^4cos^2x}$ $\quad (II) \int_0^\infty \frac{dx}{1+x^4cos^2x}$.

Solution. The integral is positive for positive value of x but the tests obtained for the convergence of positive integrands so far, are not applicable. In order to show the integral convergent we proceed as follows:

Consider $\int_0^{n\pi} \frac{xdx}{1+x^4cos^2x}$.

Therefore $\int_0^{n\pi} \frac{xdx}{1+x^4cos^2x} \;=\; \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4cos^2x}$.

Now, $\forall\; x \in [(r-1)\pi,\; r\pi]$.

We have

$$\frac{x}{1+x^4cos^2x} \;\; \geq \;\; \frac{(r-1)\pi}{1+r^4cos^2x}$$

Therefore $\int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4cos^2x} \;\; \geq \;\; \int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi dx}{1+x^4cos^2x}$

Putting $x \;=\; (r-1)\pi \;+y$, we see that

$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi dx}{1+x^4cos^2x} \quad = \quad \int_0^{\pi} \frac{(r-1)\pi dy}{1+r^4\pi^4cos^2\{(r-1)\pi+y)\}}$$

$$= \int_0^{\pi} \frac{(r-1)\pi dy}{1+r^4\pi^4cos^2y}$$

$$= 2(r\text{-}1)\pi \int_0^{\frac{\pi}{2}} \frac{dy}{1+r^4\pi^4cos^2y}$$

$$= 2(r\text{-}1)\pi \int_0^{\frac{\pi}{2}} \frac{sec^2ydy}{1++tan^2y+r^4\pi^4}$$

$$= \frac{2(r-1)\pi}{\sqrt{1+r^4\pi^4}}\ tan^{-1}\left(\frac{tany}{\sqrt{1+r^4\pi^4}}\right)\Big|_0^{\frac{\pi}{2}} \quad = \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}\ .$$

Therefore, $\sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4cos^2x} \quad \geq \quad \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$.

Hence $\lim_{n\to\infty} \int_0^{n\pi} \frac{xdx}{1+x^4cos^2x} \ \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$.

But $\sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$ is a divergent series $\left(\sim \sum_{r=1}^{n} \frac{1}{r}\right)$.

Therefore $\int_0^{\infty} \frac{xdx}{1+x^4cos^2x}$ is divergent.

(II) $\int_0^{\infty} \frac{dx}{1+x^4cos^2x}$ try yourself

Inequalities.

Definition 3.5. If $a_1, a_2, a_3, \dots \dots a_n$ are n real numbers, then their Arithmetic mean is defined as

$$A \quad = \frac{a_1+a_2+\cdots a_n}{n}$$

$$= \sum_{i=1}^{n} \frac{a_i}{n}\ .$$

If the above numbers are positive then, their Geometric mean is defined by

$$G = (a_1, a_2, a_3, \dots \dots a_n)^{\frac{1}{n}}$$

and the recipocial of the arithmetic mean of the recipocials of $a_1, a_2, a_3, \dots \dots a_n$ is defined to be the harmonic mean

$$H= \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$ where H is the Harmonic Mean.

Arithmetic Mean - Geometric mean Inequality.

Theorem 3.16. Let $a_1, a_2, a_3, \dots \dots a_n$ be n positive numbers. If A denotes their Arithmetic mean and G denotes their Geometric mean, then $A \geq G$. Equality sign holds if and only if $a_1 = a_2 = a_3 = \dots\dots = a_n$.

Proof. For $n = 1$, there is nothing to prove as $A = a_1 = G$.

For $n = 2$, we have to show that

$$\frac{a_1+a_2}{2} \geq (a_1a_2)^{1/2} .$$

We know that $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$

$$a_1 + a_2 - 2\sqrt{a_1}\sqrt{a_2} \geq 0$$

Therefore , $\frac{a_1+a_2}{2} \geq (a_1a_2)^{\frac{1}{2}}$.

Thus result holds for $n=2$ *and the equality holds if and only if* $a_1 = a_2$.

Now we used induction on n. For $n = 4 = 2^2$, we have

$$\frac{a_1+a_2+a_3+a_4}{2^2} = \frac{a_1+a_2+a_3+a_4}{4} = \frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2}$$

$$\geq \frac{(a_1a_2)^{\frac{1}{2}}+(a_3a_4)^{\frac{1}{2}}}{2}$$ (By previous case).

$$\geq \ [((a_1a_2)^{\frac{1}{2}}(a_3a_4)^{\frac{1}{2}}]^{\frac{1}{2}}$$

Thus, $\left(\frac{a_1+a_2+a_3+a_4}{4}\right) \geq (a_1a_2a_3a_4)^{\frac{1}{4}}$.

Thus result holds for $n = 4$ i.e. , 2^2.

Suppose result holds for $n = m$ that is, for 2^m .

Let n be a positive integer not of the form 2^m.

We choose k suitable, such that $2^m > n$.

Thus $2^m > n$ is a positive integer.

Let $$K = \frac{a_1 + a_2 + a_3 + \ . \ . \ . \ + a_n}{n} \tag{22}$$

$$and \quad a_{n+1} = a_{n+2} = . \ . \ . = a_{2^m} = K \tag{23}$$

Consider the product $(a_1 . a_n . a_3 \ . \ . \ a_n . a_{n+1} \ . a_{n+2} \ldots \ a_{2^m})$ which has 2^m form.

Since the inequality is supposed to be true for all positive integral powers of 2, we have

$$(a_1 . a_2 \ . \ . \ . \ a_n \ . \ . \ . a_{2^m})^{\frac{1}{2^m}} \leq \frac{a_1 + a_2 + \ldots + a_n + a_{n+1} + . \ . \ . + a_{2^m}}{2^m}$$

$$(a_1 . a_2 \ . \ . \ . \ a_n \ . \ . \ . a_{2^m}) \leq \left\{ \frac{(a_1 + a_2 + \ldots + a_n) + (a_{n+1} + . \ . \ . + a_{2^m})}{2^m} \right\}^{2^m}$$

$$= \left[\frac{nK + (2^m - n)K}{} \right]^{2^m}$$

With equality iff $a_1 = a_2 = a_3 = \ldots\ldots = a_n$.

$$\therefore \quad (a_1.a_2...a_n)\,K^{2^{m-n}} \leq K^{2^m}$$

$$\Rightarrow \quad (a_1.a_2...a_n) \leq K^n$$

$$\Rightarrow \quad (a_1.a_2...a_n)^{\frac{1}{n}} \leq K$$

or $$(a_1.a_2...a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + ... + a_n}{n}$$

with equality iff $a_1 = a_2 = a_3 = ... = a_n$.

Combing this result with the earlier one, we come to the conclusion that if $a_1, a_2, a_3, \dots\dots a_n$ are n positive numbers, then

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq (a_1.a_2...a_n)^{\frac{1}{n}}$$

or $A \geq G$.

Corollary. If $a_1, a_2, a_3, \dots\dots., a_n$ are n positive real numbers , then

$$G \geq H.$$

Proof. We know that for any positive integer n,

$$\frac{a_1+a_2+\cdots a_n}{n} \geq (a_1 a_2 a_3 \dots\dots. a_n)^{\frac{1}{n}} \text{ where } a_i > 0 \;\forall\; i, \;\; 1 \leq i \leq n\,.$$

Thus result holds for $\frac{1}{a_1}, \frac{1}{a_2}, \dots\dots \frac{1}{a_n}$.

Therefore

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots\dots\dots\dots\frac{1}{a_n}}{n} \geq (a_1 a_2 a_3 \dots\dots. a_n)^{\frac{1}{n}}.$$

$$\therefore \qquad \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \le \left(\frac{1}{a_1}.\frac{1}{a_2}.\frac{1}{a_3}...\frac{1}{a_n}\right)^{\frac{1}{n}}$$

$\therefore \quad H \le G$.

Equality holds if and only if $a_1 = a_2 = \cdots . = a_n$.

Corollary. Since $A \ge G$ and $G \ge H$, thus

$$A \ge G \ge H \text{ .}$$

This gives $\frac{a_1+a_2+\cdots a_n}{n} \quad \ge \quad \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots\ldots\ldots\ldots\frac{1}{a_n}}$.

Therefore, $\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \ge n^2$.

Theorem Cauchy Schwarz Inequality 3.17.

If $a_1, a_2, a_3, \ldots\ldots., a_n$ and $b_1, b_2, b_3, \ldots\ldots., b_n$ are real numbers, then

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \text{ .}$$

Proof. Let λ be a real number, then

$(a_i \lambda + b_i)^2 \ge 0$ *with equality iff* $a_i \lambda + b_i = 0$ *for* $i = 1,2,3,\ldots n$.

This implies $a_i^2 \lambda^2 + 2 a_i b_i \lambda + b_i^2 \ge 0$ *for* $i = 1,2,3,\ldots$

Adding for i=1, 2, 3. . . , we get

$$\sum_{i=1}^{n} a_i^2 \lambda^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)\lambda + \sum_{i=1}^{n} b_i^2 \ge 0$$

i.e., $f(\lambda) = A\lambda^2 + 2B\lambda + C \ge 0$ *for every real* λ

where $A = \sum_{i=1}^{n} a^2{}_i$ $B = \sum_{i=1}^{n} a_i b_i$ *and* $C = \sum_{i=1}^{n} b_i{}^2$.

Now obviously $A \geq o$.

If A=0, then there is nothing to prove because both sides of the proposed inequality reduce to zero.

So, let $A > o$.

We claim that $f(\lambda) \geq o \;\; for \;\; every \;\; \lambda \; , implies \;\; B^2 \leq AC$.

If this is not true, then $B^2 > AC \; , so$

$$f\left(\frac{-B}{A}\right) = \frac{AB^2}{A^2} + 2B\left(\frac{-B}{A}\right) + C = \frac{-B^2 + AC}{A} < 0 .$$

Thus, for $\lambda = \frac{-B}{A}$, $f(\lambda) < 0$ which contradicts the fact that

$f(\lambda) \geq 0 \;\; for \;\; every \;\; real \;\; \lambda$.

Hence our supposition that $B^2 > AC$ is wrong.

Therefore $B^2 \leq AC$ must be true.

$$\therefore \quad \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1} a^2{}_i\right)\left(\sum_{i=1}^{n} b_i{}^2\right)$$

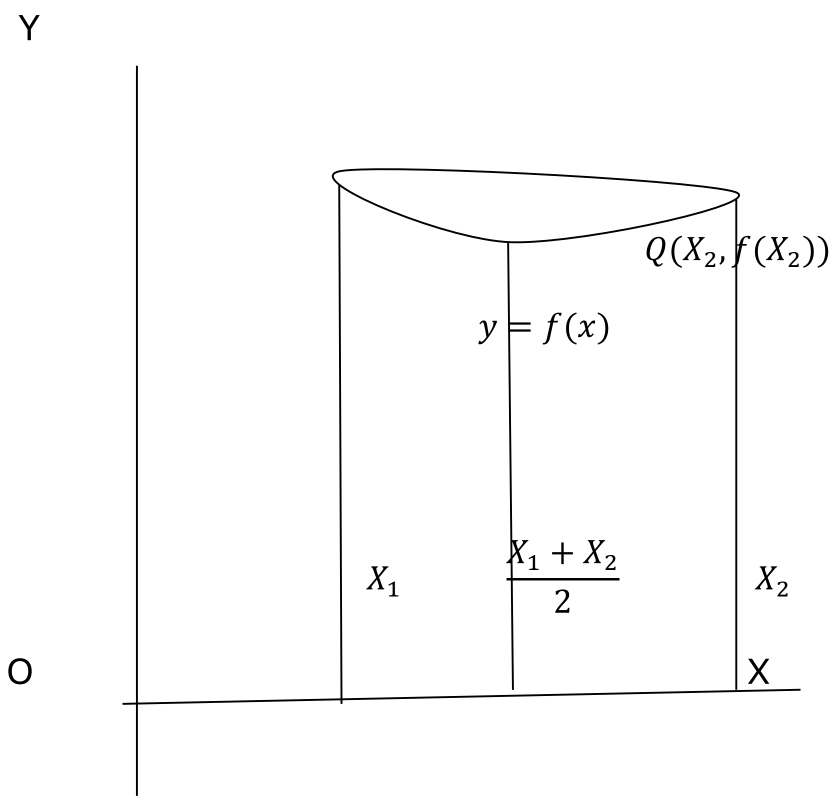
or $\quad |\sum_{i=1}^{n} a_i b_i| \;\leq\; (\sum_{i=1}^{n} a_i{}^2)^{\frac{1}{2}} (\sum_{i=1}^{n} b_i{}^2)^{\frac{1}{2}}$

Hence we have Cauchy’s - Schwarz Inequality, with equality iff

$a_i \lambda + b_i = 0 \;\; for \;\; i = 1,2,3,\ldots n.$

Convex Function.

Let $y = f(x)$ be well defined in some interval and $let\ x_1 \neq x_2\ .Say\ x_1 < x_2$ be the abscissa the graph of $y = f(x)$ so that the point P and Q are $[x_1, f(x_1)]\ \ and\ [x_2, f(x_2)]$ respectively.



If the graph of f(x) between P and Q lies below the cord PQ,then the function is said to be convex downwards or simply convex. The equation of the straight line through PQ is said to be convex downwards or simply convex.

The equation of the straight line PQ is

$$\frac{y - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

That is $\quad y = f(x_1) + \frac{x - x_1}{x_2 - X_1}[f(x_2) - f(x_1)].$

Since the point $\frac{x_1+x_2}{2}$ lies between $x_1$and x_2 and the curve is convex, the y coordinate of the curve must be less than the y coordinate of the cord i.e., we must have

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$$

Thus, $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$ for convex function f.

Theorem 3.16. Prove that if f is convex function, then

$$f\left(\frac{x_1+x_2+\cdots x_n}{n}\right) \le \frac{f(x_1)+f(x)+\cdots f(x_n)}{n} .$$

Proof. For $n = 1$, there is nothing to prove.

We will first prove the result for all positive integer $m = 2^n$ by using induction on n.

For $n = 1$, then result holds, because $f\left(\frac{x_2+x_1}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$.

For a convex function, suppose result holds for 2^k.

That is

$$f\left(\frac{x_1+x_2+\cdots x_{2^k}}{2^k}\right) \le \frac{f(x_1)+f(x_2)+\cdots f(x_{2^k})}{2^k} \qquad (24)$$

We will show that result holds for2^{k+1}, we have

$$f\left(\frac{x_1+x_2+\cdots x_{2^k}+x_{1+2^k}+\cdots x_{2^{k+1}}}{2^k}\right) \le f\left(\frac{\frac{x_1+x_2+\cdots x_{2^k}}{2^k}+\frac{x_{1+2^k}+\cdots x_{2^{k+1}}}{2^k}}{2}\right)$$

$$\le \frac{f\left(\frac{x_1+x_2+\cdots x_{2^k}}{2^k}\right)+f\left(\frac{x_{1+2^k}+\cdots x_{2^{k+1}}}{2^k}\right)}{2}$$

because result holds 2 terms.

This implies that

$$f\left(\frac{x_1+x_2+\cdots x_{2^{k+1}}}{2^{k+1}}\right) \leq \frac{\frac{f(x_1)+f(x_2)+\cdots f(x_{2^k})}{2^k}+\frac{f\left(x_{1+2^k}\right)+\cdots f(x_{2^{k+1}})}{2^k}}{2}$$ by (24) .

Therefore

$$f\left(\frac{x_1+x_2+\cdots x_{2^{k+1}}}{2^{k+1}}\right) \leq \frac{f(x_1)+f(x_2)+\cdots f\left(x_{2^{k+1}}\right)}{2.2^k}$$

$$= \frac{f(x_1)+f(x_2)+\cdots f(x_{2^{k+1}})}{2^{k+1}} .$$

Thus result holds for 2^{k+1} .

Hence by principal of Mathematical induction result holds for all integers of the form 2^n , $n \geq 1$.

Now, let n be any positive integer. Choose positive integer $m, such\ that$

$$2^m > n$$

Let $\frac{x_1+x_2+\cdots x_n}{n} = K$ (25)

$$x_{n+1} = x_{n+2} = \cdots = x_{2^m} = K \qquad (26)$$

$$f(x_1) + f(x_2) + \cdots + f(x_n) = K_1 \qquad (27)$$

We have

$$f\left(\frac{x_1+x_2+\cdots x_n+\cdots x_{2^m}}{2^m}\right) \leq \frac{f(x_1)+f(x_2)+\cdots+f(x_n)+f(x_{n+1})+\cdots f\left(x_{2^m}\right)}{2^m} .$$

This implies that

$$f\left(\frac{nK+(2^m-n)K}{2^m}\right) \leq \frac{K_1+(2^m-n)f(k)}{2^m}$$

$$f(K) \leq \frac{K_1+2^m f(K)-nf(K)}{2^m}$$

$$2^m f(K) \leq K_1 + 2^m f(K) - nf(K)$$

$$0 \leq K_1 - nf(K)$$

This implies $nf(K) \leq K_1$ *or* $f(K) \leq \frac{K_1}{n}$.

Therefore, $f\left(\frac{x_1+x_2+\cdots x_n}{n}\right) \leq \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$ for all n, not of the form 2^m.

Hence $f\left(\frac{x_1+x_2+\cdots x_n}{n}\right) \leq \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$ for all $n \geq 1$, if f is convex.

Proposition . Suppose f is defined $in\ [a,b]$ and $f''(x)$ exists, $f''(x) \geq 0$ in this interval. Then f is convex .

Proof. Let $t_1 \neq t_2$, say $t_1 < t_2$ i.e., any two points in $[a,b] = I$, so that

$$t_1 < \frac{t_1+t_2}{2} < t_2 \ .$$

Then , $f(t_1) = f\left(\frac{t_1+t_2}{2}\right) + \left(\frac{t_1-t_2}{2}\right) f'\left(\frac{t_1+t_2}{2}\right) + \frac{1}{2}\left(\frac{t_1-t_2}{2}\right)^2 f''(C_1)$

where $t_1 < C_1 < \frac{t_1+t_2}{2}$.

Also, $f(t_2) = f\left(\frac{t_1+t_2}{2}\right) + \left(\frac{t_2-t_1}{2}\right) f'\left(\frac{t_1+t_2}{2}\right) + \frac{1}{2}\left(\frac{t_2-t_1}{2}\right)^2 f''(C_2)$

where $\frac{t_1+t_2}{2} < C_2 < t_2$.

Adding these two equations, we get

$f(t_1) + f(t_2) = 2f\left(\frac{t_1+t_2}{2}\right) + E$ *where* $E = \frac{1}{2}\left(\frac{t_2-t_1}{2}\right)^2 \left[f''(C_1) + f''(C_2)\right] \geq 0.$

Therefore, $f(t_1) + f(t_2) \geq 2f\left(\frac{t_1+t_2}{2}\right)$.

This implies that $f\left(\frac{t_1+t_2}{2}\right) \leq \frac{f(t_1)+f(t_2)}{2}$.

Therefore f is convex.

Proposition. If f is convex and f'' exist in $[a,b]$, then $f'' \geq 0\ in\ [a,b]$.

Proof. Let $h > 0$

Take $x - h = x_1,\ x + h = x_2$.

Since f is convex.

Therefore $f\left(\frac{x_2+x_1}{2}\right) \le \frac{(x_1)+f(x_2)}{2}$.

This implies that

$$f\left(\frac{x-h+x+h}{2}\right) \le \frac{f(x-h)+f(x+h)}{2}$$

$$f(x) \le \frac{f(x-h)+f(x+h)}{2} .$$

This gives $f(x-h)+f(x+h)-2f(x) \ge 0$ (28)

Since $f''(x)$ exists , we have

$$\lim_{h\to 0} \frac{f(x-h)+f(x+h)-2f(x)}{h^2} = f''(x) .$$

Therefore by (28) , we have $f''(x) \ge 0.$

In the year 1906 Jenson obtained some considerable extensions of the AM-GM inequality. These extensions were based on the theory of convex functions, founded by Jenson himself.

Theorem Jensen's Inequality 3.18.

Suppose f is convex and $f''(x)$ exists finitely in [a, b] and $x_1, x_2, \ldots . x_n$ are n-points in this interval. Further let $a_1, a_2, \ldots . a_n$ be n positive numbers. Then

$$f\left(\frac{a_1x_1+a_2x_2+\cdots a_nx_n}{a_1+a_2+\cdots a_n}\right) \le \frac{a_1f(x_1)+a_2f(x_2)+\cdots a_nf(x_n)}{a_1+a_2+\cdots a_n} .$$

Proof. Since f is convex and $f''(x)$ exists finitely in [a, b], therefore $f''(x) \ge 0$.

Let $\beta = \frac{a_1x_1+a_2x_2+\cdots a_nx_n}{a_1+a_2+\cdots a_n} = \frac{\sum_{i=1}^{n} a_ix_i}{\sum_{i=1}^{n} a_i}$.

This implies that $\sum_{i=1}^{n} a_i x_i - \beta \sum_{i=1}^{n} a_i = 0$ (29)

By Taylors theorem on f(x) defined in [a,b], we have

$$f(x_i) = f(\beta) + (x_i - \beta) f'(\beta) + \frac{(x_i-\beta)^2}{2} f''(C_i)$$

where $\beta < C_i \leq x_i$ *for all* $i = 1,2,3, \ldots \ldots, n$.

Multiplying both sides by a_i, *we get*

$$a_i\ f(x_i) = a_i f(\beta) + (a_i x_i - a_i \beta) f'(\beta) + \frac{a_i (x_i-\beta)^2}{2!} f''(C_i) \text{ for } i = 1,2,3, \ldots n .$$

Since $\frac{a_i (x_i-\beta)^2}{2!} f''(C_i) \geq 0$.

Therefore, $a_i\ f(x_i) \geq a_i f(\beta) + (a_i x_i - a_i \beta) f'(\beta)$.

By adding, we get

$$\sum_{i=1}^{n} a_i\, f(x_i) \geq f(\beta) \sum_{i=1}^{n} a_i + \left(\sum_{i=1}^{n} (a_i x_i - a_i \beta) f'(\beta)\right).$$

By (29) coefficient $f'(x)$ is zero.

Therefore $\sum_{i=1}^{n} a_i\, f(x_i) \geq f(\beta) \sum_{i=1}^{n} a_i$.

This implies that $f(\beta) \leq \frac{\sum_{i=1}^{n} a_i\, f(x_i)}{\sum_{i=1}^{n} a_i}$.

This gives $f\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right) \leq \frac{\sum_{i=1}^{n} a_i\, f(x_i)}{\sum_{i=1}^{n} a_i}$ which is required Inequality.

Deduction from Jensen's Inequality.

Consider the function

$$f(x) = -\log x, \qquad x > 0 .$$

Then $$f''(x) = \frac{1}{x^2} > 0, \qquad x > 0 .$$

Therefore f is convex function for all positive x.

Let $t_1, t_2, \ldots., t_n$ be positive numbers and $a_1, a_2, \ldots. a_n$ be positive numbers. Then by Jensen's Inequality

$$f\left(\frac{a_1t_1+a_2t_2+\cdots a_nt_n}{a_1+a_2+\cdots a_n}\right) \leq \frac{a_1f(t_1)+a_2f(t_2)+\cdots a_nf(t_n)}{a_1+a_2+\cdots a_n}$$

This implies that

$$-\log x\left[\frac{a_1t_1+a_2t_2+\cdots a_nt_n}{a_1+a_2+\cdots a_n}\right] \leq \frac{-[a_1 log t_1+a_2 log t_2+\cdots+a_n \log t_n]}{a_1+a_2+\cdots a_n}$$

This implies

$$\log x\left[\frac{a_1t_1+a_2t_2+\cdots a_nt_n}{a_1+a_2+\cdots a_n}\right] \geq \log({t_1}^{a_1}.{t_2}^{a_2}\ldots.{t_n}^{a_n})^{\frac{1}{a_1+a_2+\cdots a_n}}$$

This implies that $\frac{a_1t_1+a_2t_2+\cdots a_nt_n}{a_1+a_2+\cdots a_n} \geq ({t_1}^{a_1}.{t_2}^{a_2}\ldots.{t_n}^{a_n})^{\frac{1}{a_1+a_2+\cdots a_n}}$ (30)

Set $a_1 = a_2 = \cdots a_n = I$,so we get from (30)

$$\frac{t_1+t_2+\cdots t_n}{n} \geq (t_1, t_2, \ldots, t_n)^{\frac{1}{n}}.$$

$$\therefore\quad A.M \geq G.M.$$

Holder's Inequality and Minkowski's Inequality 3.19.

If $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$ and $a_j, b_j,\ \ j = 1,2,3,\ldots,n$ are real numbers , then

$$\sum_{j=1}^{n}|a_jb_j| \leq \left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|b_j|^q\right)^{\frac{1}{q}} \qquad \text{(a)}$$

Proof. For the proof, we first prove Lemma.

Lemma. If $1 < p < \infty$ and q be such that $\frac{1}{p}+\frac{1}{q} = 1$, then for any non-negative real numbers a and b, we have

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}.$$

Proof of lemma. If $b = 0$, then there is nothing to prove.

Let then b > 0, then for $t \in R$.

Let $f(t) = \frac{1}{q}+\left(\frac{1}{p}\right)t - t^{\frac{1}{p}}$.

Then $f'(t) = \frac{1}{p}\left[1 - t^{\frac{1}{p}-1}\right]$

$= \frac{1}{p}\left[1 - t^{-\frac{1}{q}}\right]$

so that for $t < 1$, $f'(t) < 0$ $and\ for\ t > 1, f'(t) > 0.$

Also $f'(t) = 0$.

This implies $t = 1$ and $f''(t) = \frac{1}{p}\left[\frac{1}{q}\ t^{\frac{1}{q}-1}\right]$

$= \frac{1}{pq} t^{-\frac{1}{p}} = \frac{1}{pq}$ $for\ t = 1$.

That is $f''(t) > 0,$ $for\ t = 1$.

$\therefore$ f(t) has minimum value at $t = 1$.

Thus $f(t) \geq f(1)$, $\forall\ t \in R$ this implies that $\frac{1}{q} + \frac{1}{p} t - t^{\frac{1}{p}} \geq 0$.

$$\therefore \quad t^{\frac{1}{p}} \leq \frac{1}{q} + \left(\frac{1}{p}\right) t.$$

Letting $t = \frac{a^p}{b^q}$, we get

$$\left(\frac{a^p}{b^q}\right)^{\frac{1}{p}} \leq \frac{1}{q} + \left(\frac{1}{p}\right)\frac{a^p}{b^q}.$$

$$\therefore \quad \frac{a}{b^{\frac{q}{p}}} \leq \frac{1}{q} + \left(\frac{1}{p}\right)\frac{a^p}{b^q}$$

$$\Rightarrow \quad \frac{ab^q}{b^{\frac{q}{p}}} \leq \frac{b^q}{q} + \frac{a^p}{p}$$

$$\Rightarrow \quad ab^{q-\frac{q}{p}} \leq \frac{b^q}{q} + \frac{a^p}{p}$$

$\Rightarrow$ $ab' \leq \frac{b^q}{q} + \frac{a^p}{p}$ $\left(\text{because} \frac{1}{p} + \frac{1}{q} = 1 . \text{Implies that} \frac{q}{p} + 1 = q\right)$.

Therefore, $ab \leq \frac{b^q}{q} + \frac{a^p}{p}$. (31)

This completes the lemma.

Proof of the theorem.

$$\left.\begin{aligned} Let\ a &= \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \\ \beta &= \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}} \end{aligned}\right\} \qquad (32)$$

If either $\alpha = 0$ or $\beta = 0$, then both sides of the Inequality (a) are zero.

Let then $\alpha \neq 0$, $\beta \neq o$ f$or\ j = 1,2,3, \dots . n$.

Letting $a = \frac{|a_j|}{\alpha}$ and $b = \frac{|b_j|}{\beta}$, then from (31), we have

$$\left(\frac{|a_j|}{\alpha}\right)\left(\frac{|b_j|}{\beta}\right) \leq \frac{1}{P}\left(\frac{|a_j|}{\alpha}\right)^P \frac{1}{q} \left(\frac{|b_j|}{\beta}\right)^q .$$

Taking summation on both sides, we get

$$\sum_{j=1}^{n} |a_j|\,|b_j| \leq a\beta\left[\frac{\frac{1}{p}\sum_{j=1}^{n}|a_j|^p}{\alpha^p} + \frac{1}{q}\frac{\sum_{j=1}^{n}|b_j|^q}{\beta^q}\right].$$

This implies that

$$\sum_{j=1}^{n} |a_j|\,|b_j| \leq \alpha\beta\left[\frac{1}{p}.\frac{\alpha^p}{\alpha^p} + \frac{1}{q}.\frac{\beta^q}{\beta^q}\right] \quad \text{by (32) gives}$$

$$\sum_{j=1}^{n} |a_j|\,|b_j| \leq \alpha\beta\left(\frac{1}{p} + \frac{1}{q}\right) = \alpha\beta$$

$$= \left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|b_j|^q\right)^{\frac{1}{q}} .$$

This completes the Holder's Inequality.

Minkowski's Inequality 3.20.

$If\ a_j, b_j,\ j =$ $1,2,3, \dots, n$ are real numbers, and $1 < p < \infty,\ 1 < q < \infty$,

such that $\frac{1}{p} + \frac{1}{q} = 1$, $then$

$$\left(\sum_{j=1}^{n}\left|a_j+b_j\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{n}\left|a_j\right|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}\left|b_j\right|^q\right)^{\frac{1}{q}} .$$

Proof. Let $Y = \sum_{j=1}^{n}\left(\left|a_j\right|+\left|b_j\right|\right)^p$

$$= \sum_{j=1}^{n}\left(\left|a_j\right|+\left|b_j\right|\right)^{p-1}\left(\left|a_j\right|+\left|b_j\right|\right)$$

$$= \sum_{j=1}^{n}\left|a_j\right|\left(\left|a_j\right|+\left|b_j\right|\right)^{p-1}+\sum_{j=1}^{n}\left|b_j\right|\left(\left|a_j\right|+\left|b_j\right|\right)^{p-1}$$

$$\le \left(\sum_{j=1}^{n}\left|a_j\right|^p\right)^{\frac{1}{p}}.\left(\sum_{j=1}^{n}\left(\left|a_j\right|+\left|b_j\right|\right)^{(p-1)q}\right)^{\frac{1}{q}} +$$

$$+ \left(\sum_{j=1}^{n}\left|b_j\right|^p\right)^{\frac{1}{p}}.\left(\sum_{j=1}^{n}\left(\left|a_j\right|+\left|b_j\right|\right)^{(p-1)q}\right)^{\frac{1}{q}} .$$

By Holder's Inequality

$$= \left(\left(\sum_{j=1}^{n}\left|a_j\right|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}\left|b_j\right|^p\right)^{\frac{1}{p}}\right) Y^{\frac{1}{q}} \qquad (33)$$

as $(p-1)q = p$

Thus

$$\left(\sum_{j=1}^{n}\left|a_j+b_j\right|^p\right)^{\frac{1}{p}} \le Y^{\frac{1}{p}} = Y^{1-\frac{1}{q}}$$

$$= Y'.Y^{-\frac{1}{q}} \le \left(\left(\sum_{j=1}^{n}\left|a_j\right|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}\left|b_j\right|^p\right)^{\frac{1}{p}}\right)$$

$$= Y^{\frac{1}{q}}.Y^{-\frac{1}{q}} \qquad by\ (33).$$

Therefore

$$\left(\sum_{j=1}^{n}\left|a_j+b_j\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{n}\left|a_j\right|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}\left|b_j\right|^q\right)^{\frac{1}{q}}.$$

This completes the proof.

Holder's Inequality from Jensen's Inequality 3.21.

Consider a function $f(x) = x^q$, $x > 0,\ q > 1$, *then*

$$f''(x) = q(q-1)x^{q-2} > 0 .$$

Therefore f is convex functions.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots \beta_n$ be all positive, then by Jensen's Inequality

$$f\left(\frac{\alpha_1\beta_1+\alpha_2\beta_2+\cdots\alpha_n\beta_n}{\alpha_1+\alpha_2+\cdots+\alpha_n}\right) \leq \quad \frac{\alpha_1 f(\beta_1)+\alpha_2 f(\beta_2)+\cdots+\alpha_n f(\beta_n)}{\alpha_1+\alpha_2+\cdots+\alpha_n}$$

This implies that $f\left[\frac{\sum_{j=1}^{n}\alpha_j\beta_j}{\sum_{j=1}^{n}\alpha_j}\right] \leq \left(\frac{\alpha_j f(\beta_j)}{\sum_{j=1}^{n}\alpha_j}\right)$.

$$\left[\frac{\sum_{j=1}^{n}\alpha_j\beta_j}{\sum_{j=1}^{n}\alpha_j}\right]^q \leq \frac{\alpha_j{\beta_j}^q}{\sum_{j=1}^{n}\alpha_j}$$

$$\frac{\sum_{j=1}^{n}\alpha_j\beta_j}{\sum_{j=1}^{n}\alpha_j} \leq \left[\sum_{j=1}^{n}\left(\frac{\alpha_j{\beta_j}^q}{\sum_{j=1}^{n}\alpha_j}\right)\right]^{\frac{1}{q}}$$

Thus $\sum_{j=1}^{n}\alpha_j\beta_j \leq \sum_{j=1}^{n}\alpha_j\left(\sum_{j=1}^{n}\alpha_j\right)^{-\frac{1}{q}}.\sum_{j=1}^{n}\alpha_j{\beta_j}^q$.

This implies that

$$\sum_{j=1}^{n}\alpha_j\beta_j \leq \left(\sum_{j=1}^{n}\alpha_j\right)^{1/p}.\sum_{j=1}^{n}\alpha_j{\beta_j}^q \quad \text{where } \frac{1}{p}+\frac{1}{q}=1 .$$

Set $\alpha_j = {a_j}^p$, and $\alpha_j{\beta_j}^q = {b_j}^q$,so that

$${\beta_j}^q = \frac{{b_j}^q}{{a_j}^p} .$$

Therefore $\alpha_j\beta_j = {a_j}^p.\left(\frac{{b_j}^q}{{a_j}^p}\right)^{\frac{1}{q}}$

$$= {a_j}^{p-\frac{p}{q}}b_j = a_j b_j$$

Therefore $\sum_{j=1}^{n}\alpha_j\beta_j = \sum_{j=1}^{n}a_j b_j \leq \left(\sum_{j=1}^{n}{a_j}^p\right)^p\left(\sum_{j=1}^{n}{b_j}^q\right)^q$

where $\frac{1}{p}+\frac{1}{q}=1$

which is Holder's Inequality .

Problem. If $a, b, c,$ are positive and $a+b+c=1$, then prove that

$\left(\frac{1}{\boldsymbol{a}}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$, when does equality hold.

Solution. We have

$$\left(\frac{\mathbf{1}}{\boldsymbol{a}}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \frac{(1-a)(1-b)(1-c)}{abc}$$

$$= \frac{b+\boldsymbol{c}}{a}\cdot\frac{c+a}{b}\cdot\frac{a+b}{c}$$

$$\geq \frac{2\sqrt{bc}.\,2\sqrt{ac}.\,2\sqrt{ab}}{abc}$$

$$= \frac{8abc}{abc} \qquad \left(\text{ as } \frac{x+y}{2} \geq \sqrt{xy}\right.$$

Thus $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$ and equality holds if and only if

$a = b = c.$

But then $(\frac{1}{a}-1)^3 = 8$ implis

$$\left(\frac{1}{a}-1\right) = 2$$

$\therefore \quad a = \frac{1}{3}$.

Thus equality holds if and only if $a = b = c$.

Problem. Prove that the volume of the maximum rectangular parallelepiped which can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is given by } \frac{8\,abc}{3\sqrt{3}} .$$

Solution. Let the semi edges of the rectangular box be x, y, z then its volume is 8xyz.Thus, V= 8xyz .

We have to maximize V subject to the condition that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 .$$

We have

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{3}} \le \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{3} = 1 \ .$$

Thus $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{3}} \le \frac{1}{3}$$

or $$\left(\frac{x\,.\,y\,\,.\,z}{a\,.\,b\,\,.\,c}\right)^{\frac{2}{3}} \le \frac{1}{3}$$

or $$\left(\frac{x\,.\,y\,\,.\,z}{a\,.\,b\,\,.\,c}\right) \le \left(\frac{1}{27}\right)^{\frac{1}{2}}$$

or $$x\,y\,z \le \frac{abc}{3\sqrt{3}} \ .$$

Hence the required maximum volume is given by $\frac{8\,abc}{3\sqrt{3}}$.

Functions of several variables.

We already know about the functions of a single independent variable and their related concepts with regard to their limits, continuity, differentiability etc. In this unit we will be discussing the functions of several variables and their characteristic properties.

Definitions 4.1.

Consider the set of n independent variables $x_1, x_2, x_3, \ldots x_n$ and one dependent variable u, then the equation $u = f(x_1, x_2 \ldots x_n)$ denotes the functional relation and is known as a function of several variables. In this case $x_1, x_2, x_3, \ldots x_n$ are n arbitrary assigned variables, the corresponding values of the dependent variable u is determined by the function relation.

The function represented above is an explicit function but where several variables are concerned it is rarely possible to obtain an equation expressing one of the variables explicitly in terms of the others. Thus most of the functions of more than one variable are implicit functions, that is to say we are given a functional relation $\varphi(x_1, x_2 \ldots x_n) = 0$ connecting the n variables, it is not in general possible to solve this equation to find an explicit function which expresses one of these variable say x, in terms of the other n-1 variables.

Limits and Continuity of functions of two or more variables.

Let u=*f(x, y)* be a function of two independent variables x and y

which is defined in some domain $D \subset R^2$. Let $(a, b) \in D$.

We say limit $f(x, y)$ *exists as* $(x, y) \to (a, b)$ *and is equal to* l *and we write*

$$\lim_{(x,y)\to(a,b)} f(x, y) = l.$$

If given $\varepsilon > 0$, howsoever small we can find a positive number δ , such that

$$\left| f(x, y) - l \right| < \varepsilon \quad when \quad ever \quad \left| x - l \right| < \delta \; , \; \left| y - l \right| < \delta$$

We should note that (x, y) *can tend to* (a, b) in any manner i.e., along any path and the value of *f(x, y)* is independent of the path choosen joining the point (x, y) *to the point* (a, b).

Example 4.1. Let $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ *as* $(x, y) \neq (0, 0)$, *find* $\lim\limits_{(x, y) \to (a, b)} f(x, y)$.

Solution. We approach the origin (0, 0) along the path *y=mx.*

Then, $y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} f(x, mx)$$

$$= \lim_{x \to 0} \frac{2x(mx)}{x^2 + m^2 x^2}$$

$$= \lim_{x \to 0} \frac{2m}{1 + m^2} = \frac{2m}{1 + m^2} \quad which \; depends \; on \; m \; and \; is \; therefore \; different \; values \; of \; m \, .$$

Hence we conclude that $\lim\limits_{(x, y) \to (a, b)} f(x, y)$ does not exist.

Example 4.2. Let $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$ *as* $(x, y) \neq (0, 0)$, *find* $\lim\limits_{(x, y) \to (a, b)} f(x, y)$.

Solution. We approach the origin (0, 0) along the path $y = mx^2$.

Then $y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f\left(x, mx^2\right)$$

$$= \lim_{x\to 0} \frac{x\left(mx^2\right)}{x^4 + m^2x^4}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} = \frac{2m}{1+m^2} \quad \textit{which depends on m and is therefore different values of m.}$$

Hence we conclude that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 4.3. Let $f(x,y) = \dfrac{2xy^2}{x^2+y^4}$ *as* $(x,y)\neq(0,0)$, *find* $\lim_{(x,y)\to(a,b)} f(x,y)$.

Solution. We approach the origin (0, 0) along the path $y=\sqrt{mx}$.

Then $y\to 0$ *as* $x\to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f\left(x, \sqrt{mx}\right)$$

$$= \lim_{x\to 0} \frac{2x\left(mx\right)}{x^2 + m^2x^2}$$

$$= \lim_{x\to 0} \frac{2x^2m}{x^2(1+m^2)} = \frac{2m}{1+m^2} \quad \textit{which depends on m and is therefore different values of m.}$$

Hence we conclude that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 4.4. Let $f(x,y) = \dfrac{2xy}{\sqrt{x^2+y^2}}$ *as* $(x,y)\neq(0,0)$, *find* $\lim_{(x,y)\to(a,b)} f(x,y)$.

Solution. We approach the origin (0, 0) along the path $y=mx$.

Then $y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0} \frac{2x(mx)}{\sqrt{x^2 + m^2 x^2}}$$

$$= \lim_{x\to 0} \frac{2x^2 m}{\sqrt{x^2\left(1+m^2\right)}} = \lim_{x\to 0} \frac{2xm}{\sqrt{1+m^2}} = 0 \; .$$

.

This shows that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to 0. We now show

that $\lim_{(x,y)\to(a,b)} f(x,y) = 0$ *by showing that*

$$\left| f(x,y) - 0 \right| < \varepsilon \quad for \quad \left| x - 0 \right| < \delta \quad and \quad \left| y - 0 \right| < \delta$$

i.e., we show that $\left| f(x,y) \right| < \varepsilon$ *for* $\left| x \right| < \delta$ *and* $\left| y \right| < \delta$, *we have*

$$(x - y)^2 \geq 0 \quad for \quad x, y \ real$$

or $$x^2 + y^2 - 2xy \geq 0$$

$$\Rightarrow \quad 2xy \leq x^2 + y^2$$

$$\Rightarrow \quad \frac{2xy}{x^2 + y^2} \leq 1$$

$$\Rightarrow \quad \frac{2xy}{\sqrt{x^2 + y^2}} \leq \sqrt{x^2 + y^2}$$

Hence $$\left| f(x,y) \right| = \left| \frac{2xy}{\sqrt{x^2 + y^2}} \right| \leq \left| \sqrt{x^2 + y^2} \right| = \sqrt{x^2 + y^2} \qquad (i)$$

We choose $|x| = \frac{\varepsilon}{\sqrt{2}} = \delta$, $|y| = \frac{\varepsilon}{\sqrt{2}} = \delta$.

Now, by equation (i),we have

$$\left| f(x,y)-0 \right| \leq \sqrt{x^2+y^2} < \sqrt{\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon .$$

Thus, $\left| f(x,y)-0 \right| < \varepsilon$ *for* $\left| x-0 \right| < \delta = \frac{\varepsilon}{\sqrt{2}}$ *and* $\left| y-0 \right| < \delta = \frac{\varepsilon}{\sqrt{2}}$.

Therefore, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = 0$.

Definition 4.2. The limit $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y)$; $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$ are known as repeated limits whereas the limit .

The $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ is known as the simultaneous limit or a double limit.

Example 4.5. Show that for the following functions the two repeated limits exist at (0, 0) and are unequal but simultaneous limit or double limit does not exists.

(i) $f(x,y) = \frac{x-y}{x+y}$ $(x,y) \neq (0,0)$.

(ii) $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ $(x,y) \neq (0,0)$.

Solution. (i) We have

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x-y}{x+y}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{x-y}{x+y}\right\}$$

$$= \lim_{x\to 0} 1 = 1.$$

Again,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x-y}{x+y}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{x-y}{x+y}\right\}$$

$$= \lim_{y\to 0}(-1) = -1.$$

Hence $\lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y).$

Hence, the two repeated limits exist but are not equal.

To see that the simultaneous limit exists or not we approach the origin along the path $y = mx$. Then $y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0}\frac{x - mx}{x + mx}$$

$$= \lim_{x\to 0}\frac{1-m}{1+m} = \frac{1-m}{1+m}$$

which depends on m *and is therefore different values of* m .

Hence the simultaneous limit does not exist.

(ii) This is left to the student as an exercise.

Continuity of functions of two or more variables.

Let u=*f(x, y)* be a function of two independent variables x and y which is defined in some domain $D \subset R^2$. Let $(a,b) \in D$.

A function $f(x,y)$ *is said to be continuious at a point* (a,b) if given $\varepsilon > 0$, howsoever small we can find a positive number δ , such that

$$\left| f(x,y) - f(a,b) \right| < \varepsilon \quad for \quad \left| x-a \right| < \delta \ , \ \left| y-b \right| < \delta$$

In other words the function $f(x,y)$ *is said to be continuious at a point* $(a,b) \in D$ *if* $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. The function $f(x,y)$ *is said to be continuious in D* if it is continuous at all points of D.

Example 4.6. Discuss the continuity of the function

$$f(x,y) = \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2} \quad as \quad (x,y) \neq (0,\ 0)$$
$$= \ 0 \qquad as \quad (x,y) = (0,0) \quad .$$

Solution. We show the function $f(x,y)$ *is continuious at a point* $(0,0)$ by showing that

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(0,0) = 0 \ .$$

In fact we show that

$$\left| f(x,y) - f(0,0) \right| < \varepsilon \quad for \quad \left| x-0 \right| < \delta \ , \ \left| y-0 \right| < \delta$$

or $$\left| f(x,y) - f(0,0) \right| = \left| f(x,y) \right| < \varepsilon \quad for \quad \left| x \right| < \delta \ , \ \left| y \right| < \delta$$

We have $$\left| f(x,y) \right| = \left| \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2} \right| = \frac{\left| xy \right| \left| x^2 - y^2 \right|}{\left| x^2 + y^2 \right|}$$
$$\leq \frac{\left| xy \right| \left\{ \left| x \right|^2 + \left| y \right|^2 \right\}}{x^2 + y^2}$$
$$= \frac{\left| x \right| \left| y \right| \left\{ x^2 + y^2 \right\}}{x^2 + y^2} = \left| x \right| \left| y \right|$$

Thus, $\left| f(x,y) \right| \le \left| x \right| \left| y \right|$.

We choose $\delta = \sqrt{\varepsilon}$.

$\therefore \quad \left| f(x,y) \right| \le \left| x \right| \left| y \right| < \sqrt{\varepsilon}\,\sqrt{\varepsilon} = \varepsilon$

$\text{for } \left| x \right| < \delta = \sqrt{\varepsilon} \text{ and } \left| y \right| < \delta = \sqrt{\varepsilon}$.

Therefore

$$\left| f(x,y) - f(0,0) \right| < \varepsilon \quad \text{for } \left| x - 0 \right| < \delta = \sqrt{\varepsilon}\ ,\ \left| y - 0 \right| < \delta = \sqrt{\varepsilon}\ .$$

This shows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

Hence the function $f(x,y)$ *is continuious at a point* $(0,0)$.

Example 4.7. Discuss the continuity of the function

$$f(x,y) = \frac{x^2 y^2}{x^4 + y^4 - x^2 y^2} \qquad (x,y) \ne (0,0)$$
$$= 0 \qquad (x,y) = (0,0) \quad .$$

Solution. Here $f(0,0) = 0$.

To see whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists or not, we approach the origin along a path $y = mx$, then $y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx)$$
$$= \lim_{x\to 0} \frac{x^2 - m^2 x^2}{x^4 + m^4 x^4 - x^4 m^2}$$
$$= \lim_{x\to 0} \frac{m^2}{1 + m^4 - m^2}$$

which depends on m and is therefore different values of m .

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Hence the function $f(x,y)$ is discontinuous at the origin.

Example 4.8. Discuss the continuity of the following function at the origin

$$f(x,y) = x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) \quad ; \quad (x,y)\neq(0,0)$$
$$= 0 \quad ; \quad (x,y)=(0,0) \quad .$$

Solution. To see whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists or not, we approach the origin along a path $y = mx$, then $y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx)$$
$$= \lim_{x\to 0} x\sin\left(\frac{1}{mx}\right) + m\,x\sin\left(\frac{1}{x}\right)$$
$$= 0 .$$

Therefore $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists as $(x,y)\to(0,0)$ and is equal to zero.

Now

$$\left|f(x,y)- f(0,0)\right| = \left|f(x,y)-0\right| = \left|f(x,y)\right| = \left|x\sin\frac{1}{y}+ y\sin\frac{1}{x}\right|$$
$$= \left|x\right| + \left|y\right|.$$

We choose $\left|x\right| < \delta = \dfrac{\varepsilon}{2}$ *and* $\left|y\right| < \delta = \dfrac{\varepsilon}{2}$.

Therefore the function $f(x,y)$ is continuous at the origin.

Partial differential of function of two or more variables.

Let $u = f(x,y)$ be a function of two independent variables x and y which is defined in some domain $D \subset R^2$, then partial derivative of

$f(x,y)$ with respect x at a point (a,b) where $(a,b)\in D$ which is denoted by $\frac{\partial f}{\partial x}(a,b)$ or $f_x(a,b)$ is defined as

$$\frac{\partial f}{\partial x}(a,b) \text{ or } f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly the partial derivative of $f(x,y)$ with respect y at a point (a,b) where $(a,b)\in D$ which is denoted by $\frac{\partial f}{\partial y}(a,b)$ or $f_y(a,b)$ is defined as

$$\frac{\partial f}{\partial y}(a,b) \text{ or } f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

More generally if $f(x_1, x_2, \ldots x_n)$ is a function of n independent variables. Then the partial derivative of f with respect $x_i \quad i = 1,2,3,\ldots .n$ at the point $(x_1, x_2, \ldots x_n)$ is defined by

$$\frac{\partial f}{\partial x_i}(a_1, a_2, a_3, \ldots a_n) \text{ or } f_{x_i}(a_1, a_2, a_3, \ldots a_n) \text{ and is defined as}$$

$$\lim_{h\to 0} \frac{f(a_1, a_2 + \ldots a_i + h \ldots a_n) - f(a_1, a_2, a_3, \ldots a_n)}{h}$$

$$\text{for } i = 1, 2, \ldots n.$$

Example 4.9. A function
$$f(x,y) = \frac{x^3 + y^3}{x - y} \qquad (x,y) \neq (0,0)$$
$$= 0 \qquad (x,y) = (0,0)$$

Show that first order partial derivative of $f(x,y)$ at a point $(0,0)$ exists but the function $f(x,y)$ is discontinuious at a point $(0,0)$.

Solution. We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{h^3}{h} - 0}{h}$$
$$= \lim_{h \to 0} h = 0 .$$

Similarly

$$\frac{\partial f}{\partial x}(0,0) = \lim_{k \to 0} \frac{f(h,0+k) - f(0,0)}{k}$$
$$= \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$
$$= \lim_{k \to 0} \frac{-k^2 - 0}{k}$$
$$= \lim_{k \to 0} k = 0 .$$

Hence, the first order partial derivative exist at (0,0) and are both equal to zero.

To see whether $\lim_{(x,y) \to (0,0)} f(x,y)$ exists or not we approach the origin along the curve $y = x - mx^3$, *then* $y \to 0$ *as* $x \to 0$ *, and we have along this curve*

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} f\left(x, mx - mx^3\right)$$
$$= \lim_{x \to 0} \frac{x^3 + \left(x - m x^2\right)^3}{x - \left(x - m x^3\right)}$$
$$= \lim_{x \to 0} \frac{1 + \left(1 - m x^2\right)}{m} = \frac{2}{m}$$

which depends on m and is therefore different values of m .

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and hence the function $f(x,y)$ is discontinuous at the origin.

Total differentiation.

Let $u = f(x,y)$ be a function of two independent variables x and y, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

i.e.,
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are continuous functions of x an y.*

Proof. We have $u = f(x,y)$.

Let x and y receive simultaneous increments δx *and* δy respectively.

Let the corresponding increments of u be δu ,then we have

$$u + \delta u = f(x + \delta x, y + \delta y)$$

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$
$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)] \qquad (i)$$

Thus we have expressed δu as a sum of two differences in which for the first difference y remains constant at the value $y + \delta y$ and x varies from x to $x + \delta x$. In the second difference x remains constant at the value x and y varies from y to $y + \delta y$.

By Mean value theorem, we have

$$f(x + \delta x, y + \delta y) - f(x, y) = \delta_x f_x(x + \theta_1 \delta x, y + \delta y) \quad \textit{where} \quad 0 < \theta_1 < 1$$

and

$$f(x, y+\delta y) - f(x, y) = \delta_y \, f_y(x, y+\theta_2 \delta y) \quad where \quad 0 < \theta_2 < 1$$

Using this in equation (i), we get

$$\delta u = \delta_x \, f_x(x+\theta_1 \delta x, y+\delta y) + \delta_y \, f_y(x, y+\theta_2 \delta y)$$

Since f_x *and* f_y are continuous functions of x and y in the domain considered.

$$\therefore \quad \lim_{(\delta x, \delta y) \to (0,0)} f_x(x+\theta_1 \delta x, y+\delta y) = f_x(x, y)$$

$$and \quad \lim_{(\delta x, \delta y) \to (0,0)} f_y(x, y+\theta_2 \delta y) = f_y(x, y) \; .$$

Hence, we have

$$f_x(x+\theta_1 \delta x, y+\delta y) = f_x(x, y) + \varepsilon_1 \quad where \quad \varepsilon_1 \to 0 \quad as \quad (\delta x, \delta y) \to (0,0)$$

$$and \quad f_y(x, y+\theta_2 \delta y) = f_y(x, y) + \varepsilon_2 \quad where \quad \varepsilon_2 \to 0 \quad as \quad (\delta x, \delta y) \to (0,0)$$

Using this equation in (ii), we get

$$\delta f = \delta u = \delta x [f_x + \varepsilon_1] + \delta y [f_y + \varepsilon_2]$$

$$= \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \delta x \, \varepsilon_1 + \delta y \, \varepsilon_2$$

$$where \quad \varepsilon_1 \to 0 \, , \; \varepsilon_2 \to 0 \quad as \quad (\delta x, \delta y) \to (0,0) \, .$$

The principle part in the increment is called the total differential of u with respect to x and y and is denoted by du. Hence, we have

$$d\,u = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y \qquad (ii)$$

If in particular, we take u=x, then $\frac{\partial u}{\partial x}=1, \frac{\partial u}{\partial y}=0$ and so $du=\delta x$. Also

$du=dx$, since u=x. Hence $dx=\delta x$. Similarly we can have $dy=\delta y$ by putting u=y. Hence (ii) becomes

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

or

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Remark. If $u=f(x,y)$ is a function of two variables in x and y and these variables are functions of a variable t, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

and if $x=\phi(t,s)$ and $y=\phi(t,s)$, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

Mean Value Theorem. If $f(x,y), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are all continuous functions in a circular domain D of Centre (a,b) and radius large enough for the point $(a+h,b+k)$ *to be within* D, *then*

$$f(a+h,b+k)=f(a,b)+h\,f_x(a+\theta h,b+\theta k)+k\,f_y(a+\theta h,b+\theta k) \quad where \quad 0<\theta<1.$$

Proof. Consider a function $g(t)=F(a+th,b+tk)$ *where* $0\le t\le 1$.

Then $g(t)$ is a continuous function in the closed interval [0, 1] and differential in the open interval (0, 1), therefore by Lagrange's Mean Value Theorem, we have

$$g(1) - g(0) = (1-0)\, g'(\theta) \qquad where \quad 0 < t < 1 .$$

or $$g(1) = g(0) + g'(\theta) \qquad where \quad 0 < t < 1 .$$

But, $$g(1) = F(a+h, b+k)$$

Thus $$F(a+h, b+k) = F(a, b) + g'(\theta) \quad where \quad 0 < t < 1 \qquad (i)$$

Now $$g(t) = F(a+th, b+tk) = F(x, y)$$

where
$$x = a + th$$
$$y = a + tk$$

Thus $$\frac{dx}{dt} = h \quad and \quad \frac{dy}{dt} = k$$

Therefore

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= f_x(x, y)\, h + f_y(x, y)\, k$$
$$= h\, f_x(a+th, b+tk) + k\, f_y(a+th, b+tk)$$

Using this in eq. (i), we get

$$F(a+h, b+k) = F(a, b) + h\, f_x(a+\theta h, b+\theta k) + k\, f_y(a+\theta h, b+\theta k) \ where \quad 0 < \theta < 1.$$

Theorem 4.1. If $u = f(x, y)$ then $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

holds no matter what the independent variables be. In other words

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

holds regardless of whether x, y are independent or dependent variables.

Proof. Let $u = f(x, y)$ be a function of two variables x and y. If x, y are independent variables, then we have already proved that

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy .$$

Now suppose x, y are dependent variables, say

$$x = \phi(r, s) \quad y = \varphi(r, s)$$

Then

$$u = f(x, y) = f(\phi(r, s), \varphi(r, s))$$
$$= f(r, s)$$

$$\therefore \quad du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds \quad where \quad r, s \ are \ independent \ variables .$$

We show

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial u}{\partial r} dr + \frac{\partial s}{\partial s} ds$$

$$Now ,$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} . \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial r}$$

$$and \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} . \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial s}$$

Substituting the values, we get

$$\frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds = \{ \frac{\partial u}{\partial x} . \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial r} \} dr + (\frac{\partial u}{\partial x} . \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial s} \} ds$$

$$= \frac{\partial u}{\partial x}.\left\{\frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial s}.ds\right\} + \frac{\partial u}{\partial y}\left\{\frac{\partial y}{\partial r}\, dr + \frac{\partial y}{\partial s} ds\right\}$$

$$= \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

$$because \quad dx = \frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial s} ds$$

$$and \quad dy = \frac{\partial y}{\partial r}\, dr + \frac{\partial y}{\partial s} ds$$

Thus $$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

and the theorem is completely proved.

Differentiability of functions of two variables.

Let $u = f(x, y)$ be a function of two variables in x and y which is defined in some domain $D \subset R^2$.

Let $(a, b) \in D$, then

$$f(a+h, b+k) = f(a, b) + h\frac{\partial f}{\partial x}(a, b) + k\frac{\partial f}{\partial y}(a, b) + \sqrt{h^2 + k^2}\,\phi(h, k)$$

$$where \quad \phi(h, k) \to (0, 0) \text{ as } (h, k) \to (0.0).$$

Example 4.10. Discuss the continuity and differentiability of the function

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} \quad as \quad (x, y) \neq (0, 0)$$

$$= 0 \quad as \quad x = y = 0 \quad at\ the\ origin.$$

Solution. We first show that the function $f(x, y)$ is continuous at $(0, 0)$.

We approach the origin along a path $y = mx$, then

$y \to 0$ *as* $x \to 0$ *and we have along this path*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0} \frac{x\, mx}{x\sqrt{1+m^2}}$$

$$= \lim_{x\to 0} \frac{mx}{\sqrt{1+m^2}} = 0 \, .$$

This shows that $\lim_{(x,y)\to(0,0)} f(x,y)$ *as* $(x,y)\to(0,0)$ exists along the path $y = mx$ and is equal to 0.

We now show that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

We have

$$|f(x,y) - f(0,0)| = |f(x,y) - 0|$$

$$= |f(x,y)|$$

$$= \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \qquad (i)$$

Now $(x-y)^2 \geq 0$ *for* x, y *real*

or $x^2 + y^2 - 2xy \geq 0$

$\Rightarrow \quad x^2 + y^2 \geq 2xy$

$\Rightarrow \quad \dfrac{x^2+y^2}{2} \geq xy$

$\Rightarrow \quad xy \leq \dfrac{x^2+y^2}{2}$

$$\Rightarrow \qquad \frac{xy}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}}{2}$$

We choose $\delta = \sqrt{2}\,\varepsilon$ for $|x| < \delta = \sqrt{2}\,\varepsilon$ *and* $|y| < \delta = \sqrt{2}\,\varepsilon$

Now from eq. (i), we have

$$|f(x,y)| = \left|\frac{\sqrt{x^2+y^2}}{2}\right| \le \left|\frac{\sqrt{2\varepsilon^2+2\varepsilon^2}}{2}\right| = \varepsilon\,.$$

Hence $|f(x,y) - f(0,0)| < \varepsilon$ *for* $|x-0| < \varepsilon = \sqrt{2}\,\varepsilon$ *and* $|y-0| < \varepsilon = \sqrt{2}\,\varepsilon$

We now show that $f(x,y)$ is not differentiable at (0,0). $f(x,y)$ will be differentiable at (0,0) if

$$f(0+h, 0+k) = f(0,0) + h\frac{\partial f}{\partial x}(0,0) + k\frac{\partial f}{\partial y}(0,0) + \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\text{where} \quad \phi(h,k) \to (0,0) \text{ as } (h,k) \to (0.0)$$

$$\text{or} \quad f(h,k) = f(0,0) + h\frac{\partial f}{\partial x}(0,0) + k\frac{\partial f}{\partial y}(0,0) + \sqrt{h^2+k^2}\,\phi(h,k) \qquad (ii)$$

$$\text{where} \quad \phi(h,k) \to (0,0) \text{ as } (h,k) \to (0.0)$$

Now

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} = 0\,.\end{aligned}$$

Similarly

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{h \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{h \to 0} \frac{0 - 0}{k} = 0 .$$

Hence from eq. (ii), $f(x, y)$ will be differentiable at (0, 0) if

$$\frac{hk}{\sqrt{h^2 + k^2}} = 0 + h.0 + k.0 + \sqrt{h^2 + k^2}\, \phi(h, k) = \sqrt{h^2 + k^2}\, \phi(h, k)$$

where $\phi(h, k) \to 0 \; as \; (h, k) \to (0, 0)$ and hence $\phi(h, k) = \frac{hk}{h^2 + k^2}$.

Here we shall approach the origin along the path

$k = mh \quad then \quad k \to 0 \; as \; h \to 0 \,, \; we \; get$

$$\lim_{(h,k) \to (0,0)} \phi(h, k) = \lim_{x \to 0} \phi(h, mh)$$

$$= \lim_{x \to 0} \frac{h \; mh}{h^2 + k^2}$$

$$= \frac{m}{1 + m^2}$$

which depends on m and is different for different values of m. Hence $\lim_{(h,k) \to (0,0)} \phi(h, k)$ does not exist and as such $f(x, y)$ is not differentiable at (0, 0).

Second and higher order partial derivatives.

Let $u = f(x, y)$ be a function of two variables in x and y which is defined in some domain $D \subset R^2$, *then*

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \ , \ \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \ , \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \ , \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

are called second order partial derivatives of the function f.

In general $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$.

We have $\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k} = f_y(a,b)$$

Now $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x} \ f_y(a,b) = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$

$$= \lim_{h \to 0} \frac{1}{h} \{ \lim_{k \to 0} \frac{f(a+h,b+k) - f(a+h,b)}{k} - \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k} \}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{1}{hk} \{ f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b) \}$$

Thus $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \lim_{h \to 0} \lim_{k \to 0} \frac{\Delta^2 f(h,k)}{hk}$

where $\Delta^2 f(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$.

Similarly $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y} \ f_x(a,b) = \lim_{h \to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$

$$= \lim_{h \to 0} \frac{1}{k} \{ \lim_{k \to 0} \frac{f(a+h,b+k) - f(a,b+k)}{h} - \lim_{k \to 0} \frac{f(a+h,b) - f(a,b)}{h} \}$$

$$= \lim_{k \to 0} \lim_{h \to 0} \frac{1}{hk} \{ f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b) \}$$

Thus $f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \lim_{k \to 0} \lim_{h \to 0} \frac{\Delta^2 f(h,k)}{hk}$.

Since, in general

$$\lim_{h\to 0}\lim_{k\to 0}\frac{\Delta^2 f(h,k)}{hk} \neq \lim_{k\to 0}\lim_{h\to 0}\frac{\Delta^2 f(h,k)}{hk}.$$

Therefore in general $f_{xy} \neq f_{yx}$.

Example 4.11. Let
$$f(x,y) = \frac{xy\left(x^2-y^2\right)}{x^2+y^2} \quad as \quad (x,y)\neq(0,0)$$
$$= 0 \qquad as \quad x = y = 0 .$$

Show that $f_{xy} \neq f_{yx}$ *at* $(0,0)$.

Solution. We have for $(x,y)\neq(0,0)$.

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left[\frac{x^3y - xy^3}{x^2+y^2}\right]$$

$$= \frac{\left(x^2+y^2\right)\left(3x^2y-y^3\right)-\left(x^3y-xy^3\right)2x}{\left(x^2+y^2\right)^2}$$

and
$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left[\frac{x^3y - xy^3}{x^2+y^2}\right]$$

$$= \frac{\left(x^2+y^2\right)\left(x^3-3xy^2\right)-\left(x^3y-xy^3\right)2y}{\left(x^2+y^2\right)^2}$$

Also
$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{0-0}{h} = 0 .$$

and
$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0}\frac{0-0}{k} = 0 .$$

Therefore $f_{xx}(0,0) = \frac{\partial}{\partial x} f_y(0,0) = \lim_{h \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$

$$= \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} = \lim_{h \to 0} \frac{h^3}{h^3} = 1.$$

Also $f_{yy}(0,0) = \frac{\partial}{\partial y} f_x(0,0) = \lim_{k \to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$

$$= \lim_{h \to 0} \frac{-\frac{k^3}{k^2} - 0}{k} = \lim_{h \to 0} \frac{-k^3}{k^3} = -1.$$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Sufficient conditions for the equality of f_{xy} *and* f_{yx}.

We now give two theorems which we show that under what conditions f_{xy} *and* f_{yx} are equal at a certain point which are known as sufficient conditions for equality of f_{xy} *and* f_{yx}.

Theorem 4.2 (Young's Theorem).

If f_x *and* f_y exist in the neighborhood of the point (a, b) and f_x *and* f_y are differentiable at the point (a, b), then

$$f_{xy} \neq f_{yx} \text{ at } (a,b).$$

Proof. We shall prove this result by taking equal increment h for both x and y and calculate $\Delta^2 f$ in two different ways

where $\Delta^2 f = f(a+h,b+h) - f(a+h,b) - f(a,b+h) + f(a,b) \quad (\because h = k)$

Now, consider the function $H(x) = f(x, b+h) - f(x, b)$

Then $H(a+h) = f(a+h, b+h) - f(a+h, b)$ so that

$$H(a+h) - H(a) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) - f(a, b)$$

$$= \Delta^2 f \;.$$

Thus $\Delta^2 f = H(a+h) - H(a)$.

Since f_x exists in the neighborhood of the point (a, b); we apply Lagrange's mean value theorem to H(x) in the interval (a, a +h), we get

$$\Delta^2 f = H(a+h) - H(a) = h\,H^{/}(a+\theta h) \quad where \quad 0 < \theta < 1$$

$$= h\{\, f_x(a+\theta h, b+h) - f_x(a+\theta h, b)\,\} \qquad (i)$$

Since f_x is differentiable at (a,b), then

$$f_x(a+\theta h, b+h) = f_x(a, b) + \theta h\, f_{xx} + h\, f_{yx} + \sqrt{\theta^2 + h^2}\,\varepsilon_1 \quad where \quad \varepsilon_1 \to 0 \quad as \quad h \to 0$$

$$and \qquad f_x(a+\theta h, b) = f_x(a, b) + \theta h\, f_{xx} + \theta h\,\varepsilon_2 \qquad where \quad \varepsilon_2 \to 0 \quad as \quad h \to 0$$

Hence

$$f_x(a+\theta h, b+h) - f_x(a+\theta h, b) = h\, f_{yx} + h[\sqrt{\theta^2+1}\,\varepsilon_1 - \theta\,\varepsilon_2]$$

$$= h\, f_{yx} + h\,\varepsilon \qquad where \quad \varepsilon = \sqrt{\theta^2+1}\,\varepsilon_1 - \theta\,\varepsilon_2$$

$$and \quad \varepsilon \to 0 \;\; as \;\; h \to 0 .$$

Using this in eq. (i), we get

$$\Delta^2 f = h^2[\, f_{yx} + \varepsilon\,] \qquad where \quad \varepsilon \to 0 \;\; as \;\; h \to 0 .$$

and therefore

$$\lim_{h \to 0} \frac{\Delta^2 f}{h^2} = f_{yx} \quad at \quad (a,b) \qquad (ii)$$

We now consider the function

$$K(y) = f(a+h, y) - f(a, y)$$

then $\quad K(b+h) = f(a+h, b+h) - f(a, b+h)$, so that

$$K(b+h) - K(b) = f(a+h, b+h) - f(a, b+h) - f(a+h, b) + f(a,b)$$
$$= \Delta^2 f$$

By the same procedure as done earlier, we get

$$\lim_{h \to 0} \frac{\Delta^2 f}{h^2} = f_{xy} \quad at \quad (a,b) \qquad (iii)$$

From eqs. (ii) and (iii), we get

$$f_{xy} = f_{yx} \quad at \quad (a,b) \ .$$

This proves the Theorem.

Theorem 4.3 (Schwartz Theorem).

If f_x *and* f_y and f_{yx} all exist in the neighborhood of the point (a, b) and f_{yx} is continuous at the point (a, b). Then f_{xy} also exists and

$$f_{xy} \neq f_{yx} \ at \ (a,b) \ .$$

Proof. We have

$$f_{xy} = \lim_{h \to 0} \lim_{k \to 0} \frac{\Delta^2 f(h,k)}{hk}$$
$$and \qquad f_{yx} = \lim_{k \to 0} \lim_{h \to 0} \frac{\Delta^2 f(h,k)}{hk}.$$

where $\Delta^2 f(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$

Now, consider the function

$$H(a) = f(a, b+k) - f(a, b)$$

Then $H(a+h) = f(a+h, b+k) - f(a+h, b)$ so that

$$H(a+h) - H(a) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$$

$$= \Delta^2 f(h,k).$$

Thus $\Delta^2 f(h,k) = H(a+h) - H(a)$.

Since f_x exists in the neighborhood of the point (a, b); we apply Lagrange's mean value theorem to H(x) in the interval (a, a +h), we get

$$\Delta^2 f(h,k) = H(a+h) - H(a) = h\, H'(a+\theta h) \quad where \quad 0 < \theta < 1$$

$$= h\{ f_x(a+\theta h, b+k) - f_x(a+\theta h, b) \} \qquad (i)$$

Since f_{yx} exists in the neighborhood of the point (a, b); we apply Lagrange's mean value theorem again to the R.H.S of eq.(i) and get

$$\Delta^2 f(h,k) = h\{ k\, f_{yx}(a+\theta_1 h, b+\theta_2 k)\}$$

$$where \quad 0 < \theta_1 < 1 \quad \& \quad 0 < \theta_2 < 1 \qquad (ii)$$

Since f_{yx} is continuous at (a,b), then

$$\lim_{(h,k)\to(0,0)} f_{yx}(a+\theta_1 h, b+\theta_2 k) = f_{yx}(a,b)$$

$$and \quad hence \quad f_{yx}(a+\theta_1 h, b+\theta_2 k) = f_{yx}(a,b) + \varepsilon \quad where \quad \varepsilon \to 0 \quad as \quad (h,k)\to(0,0).$$

Using this in eq. (ii), we get

$$\Delta^2 f(h,k) = hk\,[f_{yx} + \varepsilon] \qquad \text{where} \quad \varepsilon \to 0 \text{ as } (h,k) \to (0,0).$$

or
$$\frac{\Delta^2 f(h,k)}{hk} = f_{yx} + \varepsilon \qquad \text{where} \quad \varepsilon \to 0 \text{ as } (h,k) \to (0,0).$$

We first take limits when $k \to 0$ *and then when* $h \to 0$, *we get*

$$\lim_{h \to 0} \lim_{k \to 0} \frac{\Delta^2 f(h,k)}{hk} = f_{yx}$$

But
$$f_{xy} = \lim_{h \to 0} \lim_{k \to 0} \frac{\Delta^2 f(h,k)}{hk}.$$

and thus $\quad f_{xy} = f_{yx}$ *at* (a,b) .

This proves the result.

Change of variables.

Let $u = f(x,y)$ be a function of two variables in x and y, then

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$d^2u = d(du) = d\left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right]$$

$$= d\left[\frac{\partial u}{\partial x} dx \right] + d\left[\frac{\partial u}{\partial y} dy \right]$$

$$= d\left(\frac{\partial u}{\partial x} \right) dx + \frac{\partial u}{\partial x} d^2x + d\left(\frac{\partial u}{\partial y} \right) dy + \frac{\partial u}{\partial y} d^2y$$

$$= \frac{\partial^2 u}{\partial x^2}(dx)^2 + \frac{\partial^2 u}{\partial y\,\partial x} dy\,dx + \frac{\partial u}{\partial x} d^2x + \frac{\partial^2 u}{\partial x\,\partial y} dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2 + \frac{\partial u}{\partial y} d^2y$$

Thus

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}dx\,dy + \frac{\partial u}{\partial x}d^2x + \frac{\partial^2 u}{\partial y^2}(dy)^2 + \frac{\partial u}{\partial y}d^2y$$

Now, if x and y are independent variables, then dx, dy are constants so that $d^2x = 0$ *and* $d^2y = 0$, *so that*

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2.$$

Example 4.12. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is invariant for change of rectangular axes.

Solution. Let the axes turn through an angle α, *then*

$$x = x' \cos\alpha - y' \sin\alpha$$

$$y = x' \sin\alpha + y' \cos\alpha$$

where (x, y) are the co-ordinates of a point with respect to XOY and (x', y') are the co-ordinates of a point with respect to $X'OY'$.

Thus, we have

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}dx\,dy + \frac{\partial u}{\partial x}d^2x + \frac{\partial^2 u}{\partial y^2}(dy)^2 + \frac{\partial u}{\partial y}d^2y \qquad (i)$$

$$d^2u = \frac{\partial^2 u}{\partial x'^2}(dx')^2 + 2\frac{\partial^2 u}{\partial x'\,\partial y'}dx'\,dy' + \frac{\partial^2 u}{\partial y'^2}(dy')^2 \qquad (ii)$$

$\because$ x', y' are independent variables so that $d^2x' = 0$ *and* $d^2y' = 0$.

Now $\quad x = x' \cos\alpha - y' \sin\alpha$

$$y = x' \sin\alpha + y' \cos\alpha$$

$$\Rightarrow \qquad dx = dx' \cos\alpha - dy' \sin\alpha$$

$$\Rightarrow \qquad dy = d\,x' \sin\alpha + d\,y' \cos\alpha$$

and $\quad d^2x = 0 \; , d^2 y = 0 .$

Putting these values in eq.(i), we get

$$d^2u = \frac{\partial^2 u}{\partial x^2}\left(dx' \cos\alpha - dy' \sin\alpha\right)^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}\left(dx' \cos\alpha - dy' \sin\alpha\right)\left(dx' \sin\alpha + dy' \cos\alpha\right)$$
$$+ \frac{\partial^2 u}{\partial y^2}\left(dx' \sin\alpha + dy' \cos\alpha\right)$$

$$= \left(\cos^2 \frac{\partial^2 u}{\partial x^2} + 2\cos\alpha \sin\alpha \frac{\partial^2 u}{\partial x \partial y} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2}\right)\left(dx'\right)^2 + \left(\sin^2 \frac{\partial^2 u}{\partial x^2} - 2\cos\alpha \sin\alpha \frac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2}\right)\left(dy'\right)^2$$
$$+ B\, dx' dy' \qquad (iii)$$

where B is the coefficient of $dx' dy'$.

Now comparing eq. 9ii) and (iii), we get

$$\frac{\partial^2 u}{\partial x'^2} = \left(\cos^2 \frac{\partial^2 u}{\partial x^2} + 2\cos\alpha \sin\alpha \frac{\partial^2 u}{\partial x \partial y} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2}\right)\left(dx'\right)^2 \qquad (iv)$$

$$\frac{\partial^2 u}{\partial y'^2} = \left(\sin^2 \frac{\partial^2 u}{\partial x^2} - 2\cos\alpha \sin\alpha \frac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2}\right)\left(dy'\right)^2 \qquad (v)$$

Adding eq's (iv) and (v), we have

$$\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} .$$

Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is invariant for the change of rectangular axes.

Maxima and Minima of function of two or more variables.

Let $u = f(x, y)$ be a function of two variables in x and y which is defined in some domain $D \subseteq R^2$.

Let $(a, b) \in D$.

If $f(x, y) \leq f(a, b)$ for all points (x, y) belonging to a neighborhood of the point (a, b), then $f(x, y)$ is said to have a relative or local maximum at (a, b) and if $f(x, y) \geq f(a, b)$ for all points (x, y) belonging to a neighborhood of the point (a, b), then $f(x, y)$ is said to have a relative or local minimum at (a, b).

At a stationary point i.e., at an extreme point, we have

$$\frac{\partial f}{\partial y} = 0 \quad and \quad \frac{\partial f}{\partial x} = 0$$

and $$d^2 f = \frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x \partial y} dx\, dy + \frac{\partial^2 f}{\partial y^2}(dy)^2 .$$

At a stationary point $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial yx}$.

Now $$d^2 f = [dx \quad dy] \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} .$$

If (a, b) is a stationary point of (x, y) then $f(x, y)$ will have a relative minima at (a, b) if $d^2 f$ is positive i.e., $d^2 f > 0$.

i.e., $d^2 f$ is a positive definite quadratic form, which is possible if

$$\frac{\partial^2 f}{\partial x^2} > 0 \quad and \quad \frac{\partial^2 f}{\partial x^2} . \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 > 0 .$$

Now, $f(x, y)$ will have a relative maxima at (a, b) if $d^2 f$ is negative i.e., $d^2 f < 0$ which is possible if

$$\frac{\partial^2 f}{\partial x^2} < 0 \quad and \quad \frac{\partial^2 f}{\partial x^2} . \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 > 0 .$$

Example 4.13. Find the maxima and minima of the function

$Z = x^2 + 3y^2 - 6y$.

Solution. At a stationary point

$$\frac{\partial Z}{\partial x} = 2x = 0 \quad and \quad \frac{\partial Z}{\partial y} = 6y - 6 = 0 \quad \Rightarrow \quad y = 1 .$$

Thus x=0 and y= 1.

Now $\frac{\partial^2 Z}{\partial x^2} = 2$, $\frac{\partial^2 Z}{\partial x \partial y} = 0$ and $\frac{\partial^2 Z}{\partial y^2} = 6$.

Thus $\frac{\partial^2 f}{\partial x^2} . \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = 2.6 - 0 = 12 > 0$.

Hence $Z = x^2 + 3y^2 - 6y$ has a minima at the point (0, 1).

Restricted Maxima and Minima.

Lagrange's method of undetermined multipliers for maxima and minima.

Let $u = f(x_1, x_2, \ldots, x_n)$ be a function of n variables which are connected with m equations of the form

$$f_1(x_1, x_2, \ldots, x_n) = 0$$

$$f_2(x_1, x_2, \ldots, x_n) = 0 \qquad \text{(I)}$$

.

$$f_m(x_1, x_2, \ldots, x_n) = 0$$

The problem is to find the stationary values of $u = f(x_1, x_2, \ldots, x_n)$ subject to m given conditions.

Lagrange's method of undetermined multipliers consists of the following

$$F(x_1, x_2, \quad x_n) = f(x_1, x_2, \ldots, x_n) + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_m f_m$$

$where \quad \lambda_1, \lambda_2, \ldots \lambda_m$ are multipliers.

The stationary point of f may be found by determining the stationary points of F.

At a stationary point of F, we have

$$\frac{\partial F}{\partial x_1} = 0 \ , \quad \frac{\partial F}{\partial x_2} = 0, \ldots \quad , \frac{\partial F}{\partial x_n} = 0$$

which gives

$$\begin{cases} \dfrac{\partial f}{\partial x_1} + \lambda_1 \dfrac{\partial f_1}{\partial x_1} + \lambda_2 \dfrac{\partial f_2}{\partial x_1} + \ . \ . \ . \ + \lambda_n \dfrac{\partial f_m}{\partial x_1} = 0 \\ \dfrac{\partial f}{\partial x_2} + \lambda_1 \dfrac{\partial f_1}{\partial x_2} + \lambda_2 \dfrac{\partial f_2}{\partial x_2} + \ . \ . \ . \ + \lambda_n \dfrac{\partial f_m}{\partial x_2} = 0 \\ . \\ . \\ . \\ \dfrac{\partial f}{\partial x_n} + \lambda_1 \dfrac{\partial f_1}{\partial x_n} + \lambda_2 \dfrac{\partial f_2}{\partial x_n} + \ . \ . \ . \ + \lambda_n \dfrac{\partial f_m}{\partial x_n} = 0 \end{cases} \qquad (II)$$

(II) are n equations out of which we shall find the value of m multipliers $\lambda_1, \lambda_2, \ ...\ \lambda_m$ are put these values in the remaining (n-m) equations of the system (II). These (n-m) equations will be free of λ's. These (n-m) equations taken together with m equations of the system (I) or in all n-m+m=n equations, which are sufficient to determine the values of $x_1, x_2, \ . \ . \ . \ , x_n$ which will give rise to the stationary values of f.

Example 4.14. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. Let x, y, z be the half of the sides of a required parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Then the volume of parallelepiped is

$$V = 2x.2y.2z = 8xyz \ .$$

We have to find the maximum value $V = 8xyz$ subject to the condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

We consider the function

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

At a stationary point of F, we have

$$\frac{\partial F}{\partial x}=8yz+\frac{2x\lambda}{a^2}=0 \qquad (i)$$

$$\frac{\partial F}{\partial y}=8xz+\frac{2y\lambda}{b^2}=0 \qquad (ii)$$

$$\frac{\partial F}{\partial z}=8xy+\frac{2z\lambda}{c^2}=0 \qquad (iii)$$

Multiplying eq. (i) by x, eq. (ii) by y and eq. (iii) by z and then adding the result, we get

$$24xyz+2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0$$

$\Rightarrow$ $$24xyz+2\lambda(1) = 0$$

or $$\lambda = -12xyz$$

Putting this value of λ in eq. (i), we get

$$8yz = \frac{2x-12xyz}{a^2}=0$$

$\Rightarrow$ $$x=\frac{a}{\sqrt{3}}$$

Similarly $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$.

Hence Volume (V) = 8xyz = $\frac{8abc}{3\sqrt{3}}$.

We show this volume is maximum by showing $\partial^2 f$ is negative.

i.e., $\partial^2 f < 0 \quad at \quad \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

At a stationary point of $(x, y, z) = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

$$\partial^2 f = \sum \frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\sum \frac{\partial^2 f}{\partial x\,\partial y} dx\ dy + \sum \frac{\partial f}{\partial x} d^2 x$$

$$= \sum \frac{2\lambda}{a^2}(dx)^2 + 2\sum 8z\ dx\ dy$$

$$= 2\lambda \sum \left(\frac{dx}{a}\right)^2 + 16\sum \frac{c}{\sqrt{3}}\ dx\ dy$$

$$= 2\lambda \sum \left(\frac{dx}{a}\right)^2 + \frac{16}{\sqrt{3}}\sum c\ dx\ dy$$

$$= 2(12\,xyz)\sum \left(\frac{dx}{a}\right)^2 + \frac{16}{\sqrt{3}}\sum c\ dx\ dy$$

$$= -\frac{8}{\sqrt{3}}abc \sum \left(\frac{dx}{a}\right)^2 + \frac{16}{\sqrt{3}}\sum c\ dx\ dy \qquad (i)$$

We have $\sum \frac{2x}{a^2} = 1$

or $\sum \frac{2x}{a^2}\,dx = 0$

and therefore at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$,we get

$$2\sum \frac{\frac{a}{\sqrt{3}}}{a^2}\, dx = 0$$

$$or \qquad \frac{2}{\sqrt{3}}\sum \frac{dx}{a} = 0 \qquad \Rightarrow \qquad \sum \frac{dx}{a} = 0 \qquad (ii)$$

Eq.(ii) gives $\left[\sum \frac{dx}{a}\right]^2 = 0$ $\qquad \left\{\because \left(\sum a\right)^2 = \sum a^2 - 12\sum ab\right\}$

$$\Rightarrow \qquad \sum\left(\frac{dx}{a}\right)^2 + 2\sum \frac{dx}{a}\frac{dy}{b} = 0$$

$$or \qquad 2\sum \frac{c\, dxdy}{abc} = -\sum\left(\frac{dx}{a}\right)^2$$

$$\Rightarrow \qquad \sum c\, dxdy = -\frac{abc}{2}\sum\left(\frac{dx}{a}\right)^2$$

Putting this in eq.(i), we get

$$d^2 f = -\frac{8abc}{\sqrt{3}}\sum\left(\frac{dx}{a}\right)^2 - \frac{16\,abc}{2\sqrt{3}}\sum\left(\frac{dx}{a}\right)^2$$

$$= -\frac{8abc}{\sqrt{3}}\sum\left(\frac{dx}{a}\right)^2 - \frac{8abc}{\sqrt{3}}\sum\left(\frac{dx}{a}\right)^2 = -\frac{16\,abc}{\sqrt{3}}\sum\left(\frac{dx}{a}\right)^2 < 0$$

i.e., $d^2 f < 0$ and therefore, maximum volume is given by $\frac{8abc}{3\sqrt{3}}$.

Jacobians.

Let $F_1, F_2, \ldots F_n$ denote n differential functions of (n+ p) variables $u_1, u_2, \ldots u_n ; x_1, x_2, \ldots x_p$, then the functional determinant

$$J\begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_n} \\ \frac{\partial F_n}{\partial u_1} & \frac{\partial F_n}{\partial u_2} & \cdots & \frac{\partial F_n}{\partial u_n}. \end{vmatrix}$$

is called the Jacobian of the n functions with respect to n variables $u_1, u_2, \ldots u_n$ and is denoted by

$$J = \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} .$$

Example 4.15. Let $x = u + v, \quad y = u - v, \ then \quad find \quad J = \frac{\partial(x, y)}{\partial(u, v)}$

Solution. We have $J = \frac{\partial(x, y)}{\partial(u, v)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Example 4.16. Let $x = u + v, \quad y = u\,v, \ then \quad find \quad J = \frac{\partial(x, y)}{\partial(u, v)}$

Solution. We have $J = \frac{\partial(x, y)}{\partial(u, v)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ v & v \end{vmatrix} = u- v.$$

Example 4.17. Let $x = r\cos\theta\ ,\quad y = r\sin\theta\ ,\ then\quad find\quad J = \dfrac{\partial(x, y)}{\partial(u, v)}$

Solution. We have $J = \dfrac{\partial(x, y)}{\partial(u, v)}$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Example 4.18. Prove that if $f(0) = 0\ ,\quad f'(x) = \dfrac{1}{1 + x^2}\ ,\ then\quad show\quad that$

$$f(x) + f(y) = f\left(\frac{x + y}{1 - xy}\right).$$

Solution. Let $u = f(x) + f(y)\quad and\quad v = \dfrac{x + y}{1 - xy}$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} f'(x) & f'(y) \\ \dfrac{1 + y^2}{(1 - xy)^2} & \dfrac{1 + x^2}{(1 - xy)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{1 + x^2} & \dfrac{1}{1 + y^2} \\ \dfrac{1 + y^2}{(1 - xy)^2} & \dfrac{1 + x^2}{(1 - xy)^2} \end{vmatrix} = 0.$$

Thus J = 0.

Therefore, there is a fundamental relation between u and v say $u = \varphi(v)$

i.e., $f(x)+f(y)=\varphi\left(\frac{x+y}{1-xy}\right).$

Put y=0, we get

$$f(x)+f(0)=\varphi(x)$$

But $f(0)=0$

Therefore $f(x)=\varphi(x) \quad \forall x.$

or $f=\varphi$

and hence $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$

Theorem 4.4. If $u_1, u_2, \ldots u_n$ are n differentiable functions of the independent variables $x_1, x_2, \ldots x_n$ and there exists an identical differentiable functional relation $f(u_1, u_2, \ldots u_n)=0$ which does not involve $x's$ explicitly then, the Jacobian $J=\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)}$ provided φ as a function of $u's$ has no stationary values in the domain considered.

Proof. We have $\varphi(u_1, u_2, \ldots u_n)=0$.

Therefore $d\varphi=0$ which implies that

$$\frac{\partial\varphi}{\partial u_1}du_1+\frac{\partial\varphi}{\partial u_2}du_2+\ldots+\frac{\partial\varphi}{\partial u_n}du_n=0 \qquad (i)$$

But $du_1=\frac{\partial u_1}{\partial x_1}dx_1+\frac{\partial u_1}{\partial x_2}du_2+\ldots+\frac{\partial u_1}{\partial x_n}du_n=0$

$$du_2=\frac{\partial u_2}{\partial x_1}dx_1+\frac{\partial u_2}{\partial x_2}du_2+\ldots+\frac{\partial u_2}{\partial x_n}du_n=0$$

.

.

$$du_n = \frac{\partial u_n}{\partial x_1} dx_1 + \frac{\partial u_n}{\partial x_2} du_2 + ... + \frac{\partial u_n}{\partial x_n} du_n = 0 \,.$$

Hence from eq. (i), we get

$$\frac{\partial \varphi}{\partial u_1}\left(\frac{\partial u_1}{\partial x_1} dx_1 + \frac{\partial u_1}{\partial x_2} du_2 + ... + \frac{\partial u_1}{\partial x_n} du_n\right) + \frac{\partial \varphi}{\partial u_2}\left(\frac{\partial u_2}{\partial x_1} dx_1 + \frac{\partial u_2}{\partial x_2} du_2 + ... + \frac{\partial u_2}{\partial x_n} du_n\right)$$

$$+ \frac{\partial \varphi}{\partial u_n}\left(\frac{\partial u_n}{\partial x_1} dx_1 + \frac{\partial u_n}{\partial x_2} du_2 + ... + \frac{\partial u_n}{\partial x_n} du_n\right) = 0$$

$$or \quad \left(\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial x_1} + ... + \frac{\partial \varphi}{\partial u_n}\frac{\partial u_n}{\partial x_n}\right) dx_1 + \left(\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial x_2} + ... + \frac{\partial \varphi}{\partial u_n}\frac{\partial u_n}{\partial x_2}\right) dx_2 + ...$$

$$+ \left(\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial x_n} + ... + \frac{\partial \varphi}{\partial u_n}\frac{\partial u_n}{\partial x_n}\right) dx_n = 0 \qquad (ii)$$

Since $dx_1, dx_2, ..., dx_n$ are arbitrary differentials of independent variables, it follows from eq. (ii) that

$$\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial x_1} + ... + \frac{\partial \varphi}{\partial u_n}\frac{\partial u_n}{\partial x_n} = 0$$

$$\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial x_2} + ... + \frac{\partial \varphi}{\partial u_n}\frac{\partial u_n}{\partial x_2} = 0$$

.

.

$$\frac{\partial \varphi}{\partial u_1}\frac{\partial u_1}{\partial x_n} + ... + \frac{\partial \varphi}{\partial u_n}\frac{\partial u_n}{\partial x_n} \qquad (iii)$$

Since φ has no stationary values in the domain considered, therefore

$$\frac{\partial \varphi}{\partial u_1} = 0 , \quad \frac{\partial \varphi}{\partial u_2} = 0 , \; . \; . \; . \; . \; , \quad \frac{\partial \varphi}{\partial u_n} = 0 \; .$$

Eliminating $\frac{\partial \varphi}{\partial u_1} , \frac{\partial \varphi}{\partial u_2} , \; . \; . \; . \; . \; , \frac{\partial \varphi}{\partial u_n}$ from the systems of equations (iii), we get

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & . \; . \; . & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & . \; . \; . & \frac{\partial u_n}{\partial x_2} \\ . & & & \\ . & & & \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & . \; . \; . & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & . \; . \; . & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & . \; . \; . & \frac{\partial u_2}{\partial x_n} \\ . & & & \\ . & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & . \; . \; . & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

i.e.,

$$J = \frac{\partial (u_1 , u_2 , \; . \; . \; . \; , u_n)}{\partial (x_1 , x_2 , \; . \; . \; . \; , x_n)} = 0 \; .$$